Algebraic constructions in residuated lattices and their applications

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Residuated structures play an important role in the field of algebraic logic since they constitue the equivalent algebraic semantics, in the sense of Blok and Pigozzi, of substructural logics (see [1, 4]). These encompass many of the interesting nonclassical logics: intuitionistic logic, fuzzy logics, relevance logics, linear logics and also classical logic as a limit case. Thus, the algebraic investigation of residuated lattices is a powerful tool in the systematic and comparative study of such logics.

In more detail, a *residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \backslash, /, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 2, 0)$ where: $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and the residuation law holds, i.e. $x \cdot y \leq z$ if and only if $y \leq x\backslash z$ if and only if $x \leq z/y$ for any $x, y, z \in A$. A residuated lattice is called *commutative* if the monoidal operation is commutative and *integral* if 1 is the top element of the lattice. Moreover, a residuated lattice is called *semilinear* if it is a subdirect product of totally ordered ones. All the above mentioned classe of residuated lattices form a variety which we will denote by: RL, CRL, IRL and SemRL respectively. An element x of a residuated lattice $\mathbf{A}$ is said to be *conical* if for any $y \in A$, $x \leq y$ or $y \leq x$; we call a residuated lattice *conical* if the unit 1 is a conical element.

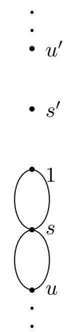
Although residuated lattices have been extensively studied over the past century, many significant subclasses still lack an effective algebraic characterization. Hence, the main goals of this contribution are:

- take a step further in providing an effective algebraic characterization for some varieties of residuated lattices;

- use the previous results to study the amalgamation property for such varieties of residuated lattices.

To be more precise, in the first part we will focus on integral residuated lattices and in particular on the class of *MTL-algebras*, the equivalent algebraic semantics of the t-norms based logic introduced by Esteva and Godo ([2]), providing a generalization of the *partial gluing construction* introduced in [6], which we will call *blockwise gluing*. Such algebraic construction is of utter importance since it allows one to find countably many failures of the amalgamation property in the following varieties of residuated lattices: SemRL, SemIRL, SemRL([7]), SemCRL ([3]), SemFL, $\mathsf{SemFL_e}$ ([3]), MTL, n-potent MTL with $n \geq 2$, IMTL and SMTL, where FL denotes the equivalent algebraic semantics of the Full-Lambek calculus and $\mathsf{FL_e}$ is its commutative version.

In the second part, we will move the focus on a special subvariety of semiconical residuated lattices, i.e. the one such that its generators are conical residuated lattices having a Sugihara chain as skeleton and the positive and negative cones are made of bubbles having an idempotent element at the top and the ones of the positive cone are just singletons. To help the intuition, we present the following figure:



where s, s', u, u' are Sugihara elements. We will call this variety S. This is a generalization of the work of Galatos and Raftery ([5]), where they analyzed the semilinear and idempotent subvariety of S. In particular, we give a description of the generators of S and we axiomatized the variety. Moreover, we show that S is completely determined by the variety (S^-, γ), i.e. the one generated by the negative cones of the generators of S plus a nucler operator γ sending every element x of an algebra to the top of its bubble. Such categorical equivalence and the description of the finitely subdirectly irreducible members of S, give us all the ingredients to find some sufficient conditions for a subvariety of S to have the amalgamation property.

The results presented in this contribution come form joint works with Nick Galatos, Francesco Manfucci and Sara Ugolini.

References

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