Congruence systems, NU terms and Global Subdirect Products

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A *congruence system* on an algebra $\mathbf{A}$ is a system of congruence equations of the form

$$\begin{cases} x \equiv a_1 \pmod{\theta_1} \\ \vdots \\ x \equiv a_n \pmod{\theta_n} \end{cases}$$

where $\theta_1, \dots, \theta_n$ are congruences of $\mathbf{A}$ and $a_1, \dots, a_n \in A$ satisfy the compatibility condition

$$a_i \equiv a_j \pmod{\theta_i \vee \theta_j} \quad \text{for all } i, j.$$

A *solution* to such a system is an element $a \in A$ satisfying $a \equiv a_i \pmod{\theta_i}$ for all i. A classical result of Baker and Pixley states that if $\mathbf{A}$ has a $(d+1)$-ary near-unanimity term, then every congruence system whose subsystems of d equations are solvable is itself solvable. This may fail if the system involves infinitely many congruences. We show how the Baker–Pixley result can be generalized to infinite systems by considering suitable compactness properties.

Global subdirect products are a special kind of subdirect product in which the index set is equipped with a topology allowing the algebra to be patched over it. A particularly important instance of a global subdirect product is the Boolean product.

We show that global subdirect representations can be characterized in terms of (possibly infinite) congruence systems. This characterization together with the above generalization of the Baker–Pixley theorem produces a uniform method to obtain both known and new global/Boolean representation theorems.