TROPICAL GEOMETRY, LECTURE 9

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- Recall $\text{trop}(V(I)) \subseteq \mathbb{R}^n$ for $I \subseteq K[T^n]$.
- Have seen: $\text{trop}(V(I))$ it is a finite intersection of tropical hypersurfaces (existence of finite tropical bases).
- Also: $\text{trop}(V(I)) \subseteq v(V(I))$.
- Example. $I = \langle x+y+1, x+2y\rangle$ over $\mathbb{C}\{\{t\}\}$ gives $\text{trop}(V(I)) = \{(0,0)\} \neq \text{trop}(V(x+y+1), x+2y)$. A finite intersection of tropical hypersurfaces is called a *tropical pre-variety.*
- Theorem (*fundamental theorem of tropical geometry*): Let K be algebraically closed with a nontrivial valuation, $I \subseteq K[T^n]$ and $X = V(I) \subseteq K^n$. Then the following are equal:
  1. $\text{trop}(V(I))$
  2. $\{w \in \mathbb{R}^n \mid \text{in}_w I \neq \langle 1\rangle\}$, and
  3. the topological closure of $v(V(I))$.

  [The first two are clearly equal: if $\text{in}_w I$ is not $\langle 1\rangle$, then $\text{in}_w f$ is not a monomial for any $f \in I$, and hence $w \in \text{trop}(V(f))$ for each $f \in I$, i.e., $w \in \text{trop}(V(I))$. For the converse we note that if $\text{in}_w I$ contains a monomial, then there is an $f \in I$ with $\text{in}_w f$ monomial.

  The third is included in the earlier two by the remarks above. For the opposite inclusion we will have to work.]
- Proposition: Let K be a field with a (possibly trivial) valuation, and let L be a valued field extension. Let $I \subseteq K[T^n]$ and consider $I' := L \otimes_K I \subseteq L[T^n]$. Then $\text{trop}(V(I)) = \text{trop}(V(I'))$.

  [$\supseteq$ is clear since the set on the left is an intersection over a larger domain. For the converse, suppose that $\text{in}_w I'$ contains a monomial. This means that there are $f_1, \ldots, f_s \in I$ and coefficients $c_1, \ldots, c_s \in L$ such that

$$\text{in}_w \sum_i c_i f_i = x^\alpha$$

  This can be read as saying that a system of K-linear equations for the c_i has an approximate solution over L. The following lemma, applied to
  1. the coefficient matrix $f_1, \ldots, f_s$, r equal to the number of monomials appearing in at least one f_i;
  2. $u_\alpha := -w \cdot \alpha$ for such a monomial x^α; and
  3. $b = (1, 0, \ldots, 0)^T$ where the 1 is on the position corresponding to the monomial x^0;

  shows that it then also has an approximate solution over K, i.e., $\text{in}_w I \ni x^\alpha$.]
- Lemma: let $A \in K^{r\times s}$, $u \in \mathbb{R}^r$, and $b \in K^r$. Suppose that there exists a row vector $z \in L^s$ such that $v((Az-b)_i) > u_i$ for all i. Then a z with this property exists in K^s.

[We have already see a version of this argument, but let's give a slightly different version. As the statement only concerns the range of A, we may assume that $A : K^s \to K^r$ is injective. In particular, we have $r \geq s$, and we prove the lemma by induction on r. For $r = s$ the matrix A is invertible, so even an exact solution to $Az = b$ exists over K. Now suppose that the statement is true for $r - 1$, which is at least s. Denote the rows of A by $a_1, \ldots, a_r \in (K^s)^*$. As $r > s$, there exists a linear relation $\sum_i \lambda_i a_i = 0$ where not all λ_i are 0. The existence of z in the lemma yields

$$\begin{aligned} v\left(\sum_i \lambda_i b_i\right) &= v\left(\sum_i \lambda_i(b_i - a_i z) + \sum_i \lambda_i a_i z\right) \\ &= v\left(\sum_i \lambda_i(b_i - a_i z) + 0\right) \\ &> \min_i (v(\lambda_i) + u_i). \end{aligned} \tag{1}$$

After rearranging the rows of A we may assume that the latter minimum is attained in $i = r$, and by multiplying all λ_i with $1/\lambda_r$ we may assume that $\lambda_r = 1$. By the induction hypothesis, there exists a $z \in K^s$ such that $v(a_i z - b_i) > u_i$ for all $i = 1, \ldots, r-1$. For this same z we have

$$\begin{aligned} v(a_r z - b_r) &= v\left(-\sum_{i=1}^{r-1} \lambda_i a_i z - b_r\right) \\ &= v\left(-\sum_{i=1}^{r-1} \lambda_i (a_i z - b_i) - \sum_{i=1}^{r} \lambda_i b_i\right) \\ &\geq \min\left\{ v\left(\sum_{i=1}^{r-1} \lambda_i (a_i z - b_i)\right), v\left(\sum_{i=1}^{r} \lambda_i b_i\right)\right\} \\ &> \min_{i=1,\ldots,r} (v(\lambda_i) + u_i), \end{aligned}$$

where the last inequality follows from (1) and the assumption on z. By assumption, the last minimum is attained in $i = r$, and equal to $v(1) + u_r = u_r$.]

- The proposition allows us to work over suitable valued field extensions of K, such as $K((\mathbb{R}))$. In particular, in the proof of the fundamental theorem, we may use that $v : K \to \mathbb{R}$ is surjective and has a section, in addition to K being algebraically closed. Thus our work from Chapter 2 becomes useful.
- In fact, $\operatorname{trop}(V(I))$ is a finite union of $v(K^*)$-rational polyhedra, and since $v(K^*)$ is divisible and non-zero (as K is algebraically closed and v non-trivial), the set of $v(K^*)$-rational points in $\operatorname{trop}(V(I))$ is dense. So for the last inclusion in the theorem it suffices to prove that if $w \in \operatorname{Trop}(V(I)) \cap (v(K^*))^n$, then there is an $x \in V(I)$ with $v(x) = w$.
- The proof of the fundamental theorem will go via projections to the hypersurface case.
- Proposition: Fix a subvariety $X \subseteq T^n$ and $m \geq \dim(X)$. Then there exists a homomorphism $\psi : T^n \to T^m$ such that $\psi(X)$ is Zariski-closed in T^m and has dimension equal to $\dim(X)$. Moreover, ψ can be chosen such that

$\ker \operatorname{trop}(\psi)$ intersects a given finite collection of m-dimensional subspaces in $\mathbb{R}^n$ trivially.

[It suffices to prove the case where $m = n-1$. Fix $l \gg 0$. Consider first the automorphism $\phi : T^n \to T^n$ defined dually by $\phi^* x_i = x_i x_n^{l^i}$ for $i = 1, \ldots, n-1$ and $\phi^* x_n = x_n$.

This maps a fixed monomial x^α to $x_1^{\alpha_1} \cdots x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n + \sum_{i=1}^{l} \alpha_i l^i}$. Thus, if we have a finite set S of monomials x^α with $\alpha \in \mathbb{Z}^n_{\geq 0}$ and we take l larger than any α_i for any $x^\alpha \in S$, then the exponents of x_n in $\phi m, \phi m'$ are distinct for $m, m' \in S$ distinct.

Apply this reasoning to the monomials appearing in a finite generating set in $K[x_1, \ldots, x_n]$ of the ideal $I = I_{T^n}(X)$, which is not the zero ideal since $\dim X < n$. This gives that, for $l \gg 0$, $\phi^*(I)$ is generated by a nonempty set of polynomials in $K[x_1, \ldots, x_n]$ whose coefficients, when regarded as polynomials in x_n, are a constant times a monomial in $x_1, \ldots, x_{n-1}$. Replace I by this $\phi^*(I)$.

We claim that we may now take for $\psi : T^n \to T^{n-1}$ the projection on the first $n-1$ coordinates. The ideal of $Y := \overline{\phi(X)}$ is then $I \cap K[T^{n-1}]$, and $K[X]$ is integral over $K[Y]$ since any one of the generators of I gives a monic equation for x_n over $K[Y]$ (the coefficient of the highest power of x_n is a monomial and hence invertible in $K[T^n]$). This shows that the map $X \to Y$ is closed and $\dim Y = \dim X$.

Composing this projection with the ϕ above yields the map ψ whose tropicalisation is $(x_1, \ldots, x_n) \mapsto (x_1 + l^1 x_n, \ldots, x_{n-1} + l^{n-1} x_n)$ with kernel spanned by $(l^1, l^2, \ldots, l^{n-1}, -1)$. For this kernel to intersect a given hyperplane with equation $\sum_i a_i x_i = 0$ trivially, we need to choose l not a root of the polynomial equation $\sum_{i=1}^{n-1} a_i t^i = a_n$. Again, this can be arranged by taking l sufficiently large.]

- By virtue of Chapter 2, the set $\{w \in \mathbb{R}^n \mid \mathrm{in}_w I \neq \langle 1 \rangle\}$ is the support of a subcomplex of the Gröbner complex of I_{proj} via the identification $\mathbb{R}^n \to \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1},\ w \mapsto (0, w)$.
- Proposition: the cells $\overline{C_{I_{\mathrm{proj}}}[w]}$ contained in this subcomplex have dimension at most the Krull dimension of $V(I)$.

[Let P be that cell, with w in its relative interior. Its affine span is $w + L$ with $L \subseteq \mathbb{R}^n$ a vector space defined over $\mathbb{Q}$. After a torus automorphism, we may assume that $L = \langle e_1, \ldots, e_k \rangle$. For $u \in L \cap \mathbb{Z}^k$ we have $\mathrm{in}_u \mathrm{in}_w I = \mathrm{in}_{w+\epsilon u} I = \mathrm{in}_w I$ for $\epsilon > 0$ sufficiently small.

Taking $u = e_i$ for $i = 1, \ldots, k$, this means that $\mathrm{in}_w I$ is homogeneous with respect to the grading in which variable x_i has degree 1 and all other variables have degree 0. In other words, it is homogeneous w.r.t. the corresponding $\mathbb{Z}^k$-grading.

Hence $\mathrm{in}_w I$ is generated by polynomials $f_1, \ldots, f_s$ that each are of the form some $m_j g_j$ where m_j is a monomial and g_j does not involve $x_1, \ldots, x_k$. Since m_j are units, these may be taken 1. This means that, for each point in $V(\mathrm{in}_w I)$, also all other points with the same last $n-k$ coordinates are in this variety. Hence, since $\mathrm{in}_w I \neq \langle 1 \rangle$ implies that $V(\mathrm{in}_w I)$ is nonempty, the Krull dimension of $k[T^n]/\mathrm{in}_w I$ is at least k, and the same holds for $k[\mathbb{A}^n]/\mathrm{in}_w I_{\mathrm{aff}}$. As we have seen, this is the image of $k[\mathbb{P}^n]/\mathrm{in}_{(0,w)} I_{\mathrm{proj}}$ under the map sending x_0 to 1. The Krull dimension of the latter ring equals that of $K[\mathbb{P}^n]/I_{\text{proj}}$, which is the dimension of $V(I)$ plus 1. The map $x_0 \mapsto 1$ corresponds to intersecting with a hyperplane, and lowers the Krull dimensional by at least 1. (More precisely: it lowers the dimension of the components of $V(\text{in}_{(0,w)} I_{\text{proj}})$ that intersect $\mathbb{A}_k^n$ in a nonempty set by 1, and removes the components that do not intersect $\mathbb{A}_k^n$.) Hence we find that, indeed, the Krull dimension of $V(I)$ is at least k.]

- To prove the fundamental theorem, we still need to prove that $\text{in}_w I \neq \langle 1 \rangle$ implies the existence of a point $x \in V(I)$ with $v(x) = w$. There is an easy reduction to the case where I is prime.
- So, now assume that I is prime, and let $d = \dim X$. We do induction on $n - d$. For $n - d = 1$ the fundamental theorem is Kapranov's theorem. Suppose that $n - d > 1$ and that the theorem holds for smaller codimension.
- The set $\text{Trop}(V(I))$ is the support of a polyhedral complex of dimension $\leq d$. For each cell P, let P_L be the linear span of $P - w$, a vector space of dimension $\leq d + 1 < n$. By the work earlier this morning, there is a torus homomorphism $\psi : T^n \to T^{n-1}$ that has the following properties:
  (1) $Y := \psi(X)$ is closed and of dimension d.
  (2) $\ker \text{trop}(\psi)$ intersects each space P_L trivially.
- Now $\text{trop}(\psi)$ maps $\text{trop}(X)$ into $\text{trop}(Y)$, so, by the induction hypothesis applied to Y, there is a point $y \in Y$ with $v(y) = \text{trop}(\psi)(w)$.
- There is a point $x \in X$ with $\psi(x) = y$. This means that $\text{trop}(\psi)v(x) = v(y) = \text{trop}(\psi)(w)$. This means that $w - v(x) \in \ker \text{trop}(\psi)$, and since this lies in L_P where P is such that $v(x) \in P$, our second property of ψ implies that $v(x) = w$.