TROPICAL GEOMETRY, LECTURE 8

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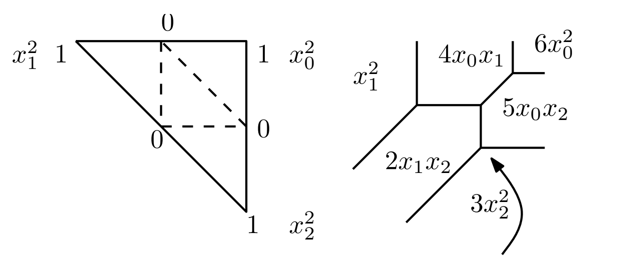
1. Rest of §2.5

- Same assumptions as last time: $v : K^* \to \mathbb{R}$ surjective and with a section $w \mapsto t^w$.
- Last time, for a homogeneous ideal $I \subseteq S_K = K[x_0, \ldots, x_n]$ and a weight vector $w \in \mathbb{R}^{n+1}$, we defined the set $C_I[w] := \{z \mid \mathrm{in}_z I = \mathrm{in}_w I\}$. We saw that this was the relative interior of an $\mathbb{R}$-rational polyhedron.
- More specifically, if w' is such that $\mathrm{in}_{w'} I$ is monomial, then there is a unique minimal set of monomials x^{α_i}, $i = 1, \ldots, s$ generating it, and there are unique polynomials $g_i \in I$ containing x^{α_i} as a term and otherwise containing only monomials not in $\mathrm{in}_{w'} I$. We saw that
$$C_I[w'] = \{z \in \mathbb{R}^{n+1} \mid \forall i = 1, \ldots, s : \mathrm{in}_z g_i = x^{\alpha_i}.$$
Its closure is the set of weight vectors z for which $\mathrm{in}_z g_i$ contains the monomial x^{α_i} for all i. Furthermore, if w, u are such that $\mathrm{in}_u \mathrm{in}_w I = \mathrm{in}_{w'} I$, then $\overline{C_I[w]}$ is a face of $\overline{C_I[w']}$ whose relative interior $C_I[w]$ equals
$$C_I[w] = \{z \in \mathbb{R}^{n+1} \mid \forall i = 1, \ldots, s : \mathrm{in}_z g_i = \mathrm{in}_w g_i\}.$$
Its closure is the set of z for which each $\mathrm{in}_z g_i$ contains the terms of $\mathrm{in}_w g_i$.
- We remark that if $\mathrm{in}_{w'} I$ is monomial, then $C_I[w']$ is open, and hence its closure is a polyhedron of full dimension $n + 1$. For distinct values of w' these polyhedra either coincide or their interiors are disjoint. By the above, all $\overline{C_I[w]}$ are faces of one of these polyhedra.
- Theorem. The sets $\overline{C_I[w]}$ as w ranges over $\mathbb{R}^{n+1}$ form a (finite) polyhedral complex whose support is all of $\mathbb{R}^{n+1}$.
- This complex, denoted $\Sigma(I)$, is called the *Gröbner complex* of I.
- The finiteness follows from the fact that each $\overline{C_I[w]}$ is a face of some $\overline{C_I[w']}$ with $\mathrm{in}_{w'} I$ monomial, and the finitude of the monomial ideals of this form.
- We first show that each face of $\overline{C_I[w]}$ is of the form $\overline{C_I[y]}$. Thus let F be a face of $\overline{C_I[w]}$ and let y be in the relative interior of F. We claim that $F = \overline{C_I[y]}$. Let w' be such that $\mathrm{in}_{w'} I$ is monomial and such that $\overline{C_I[w]}$ is a face of $P := \overline{C_I[w']}$. Then we can choose a $u \in \mathbb{R}^{n+1}$ and $\epsilon > 0$ sufficiently small such that both $w + \epsilon u$ and $y + \epsilon u$ lie in the open polyhedron $C_I[w']$. By the above, $\overline{C_I[y]}$ is the (unique) face of P whose relative interior contains y. By assumption, it is contained in the other face $\overline{C_I[w]}$, hence it is a face of the latter, and hence equal to the unique face F that has y in its relative interior.
- We next show that two polyhedra $P := \overline{C_I[w]}$ and $Q := \overline{C_I[z]}$ either have $P \cap Q = \emptyset$ or else their intersection is a face of both. Suppose that the intersection is non-empty. Since $\mathrm{in}_y I$ takes only finitely many values for

$y \in P$ (corresponding to the faces of P, see above), there is one value whose pre-image in $P \cap Q$ is dense. Take y in that pre-image. Then by construction, $P \cap Q$ is contained in $\overline{C_I[y]}$. We claim that equality holds. Let $P' = \overline{C_I[w']}$ be full-dimensional and having P as a face. Then $\overline{C_I[y]}$ is the unique face of P' containing y in its relative interior, and the face P of P' contains y. Then P must contain $\overline{C_I[y]}$. Similarly for Q.

- Example 2.5.9 from the book: the 19 initial ideals of $\langle tx_1^2 + 2x_1x_2 + 3tx_2^2 + 4x_0x_1 + 5x_0x_2 + 6tx_0^2 \rangle$.



$2x_2 + 1 = x_1 + x_2 = x_0 + x_2$
$x_0 = 0$ gives $(0, 0, -1)$

- Definition: a subset $G \subseteq I$ is called a *universal Gröbner basis* if it is finite and $\mathrm{in}_w(G)$ generates $\mathrm{in}_w I$ for all $w \in \mathbb{R}^{n+1}$.
- Theorem: any homogeneous ideal I has a universal Gröbner basis. [Indeed, for each of the finitely many monomial initial ideals $\mathrm{in}_w I$, take the corresponding set $g_1, \ldots, g_s \in I$ as above.]

2. Section 2.6—tropical bases

- In the last weeks, have concentrated on homogeneous ideals in $K[x_0, \ldots, x_n]$. This was crucial for various of the proofs, involving Hilbert functions.
- But tropical varieties most naturally arise from ideals $I \subseteq K[T^n]$, i.e., in a Laurent polynomial ring.
- The definitions of $\mathrm{in}_w f$ and $\mathrm{in}_w I$ carry over to this situation, and their properties are similar, e.g. $\mathrm{in}_w(fg) = \mathrm{in}_w(f)\mathrm{in}_w(g)$, and $\mathrm{in}_w I$, while defined as the ideal *generated* by all initial forms of elements in I, is in fact equal to (zero and) the set of all initial forms of elements of I.
- But note that as soon as $\mathrm{in}_w f$ is a monomial for some $f \in I$, then $\mathrm{in}_w I$ is all of $k[T^n]$. We'll be interested in the case where this does *not* happen.
- We'll consider $I_{\mathrm{proj}} := (I_{\mathrm{aff}})_{\mathrm{proj}}$, a homogeneous ideal in $K[x_0, \ldots, x_n]$.
- Lemma: under mapping x_0 to 1, $\mathrm{in}_{(0,w)} I_{\mathrm{proj}}$ is mapped onto $\mathrm{in}_w I_{\mathrm{aff}}$. Moreover, every element of $\mathrm{in}_w I$ is of the form $x^\alpha f$ where $f = g(1, x_1, \ldots, x_n)$ for some $g \in \mathrm{in}_{(0,w)} I_{\mathrm{proj}}$.

  [Let $f \in K[x_1, \ldots, x_n]$ and let $g := x_0^d f(x_1/x_0, \ldots, x_n/x_0)$ be its homogenisation. First note that $\mathrm{trop}(g)(0, w) = \mathrm{trop}(f)(w) =: W$. Then

$$\mathrm{in}_{(0,w)} g = \overline{t^{-W} g(t^0 x_0, t^{w_1} x_1, \ldots, t^{w_n} x_n)} = \overline{x_0^d f(t^{w_1} x_1/x_0, \ldots, t^{w_n} x_n/x_0)}$$

  and substituting $x_0 = 1$ yields $\mathrm{in}_w(f)$.

Letting f range over I_{aff}, we find that the image of $\text{in}_{(0,w)} I_{\text{proj}}$ contains $\text{in}_w I_{\text{aff}}$. Conversely, any $g \in I_{\text{proj}}$ equals some power of x_0 times the homogenisation of $f := g(1, x_1, \ldots, x_n) \in I_{\text{aff}}$, hence $\text{in}_{(0,w)}$ is mapped into $\text{in}_w I_{\text{aff}}$ by sending x_0 to 1.

The last sentence follows because each element of I is of the form $x^\alpha f$ for some $f \in I_{\text{aff}}$.]

- Useful property of initial ideals: if $ini_u(\text{in}_w(I)) = \text{in}_w(I)$ for some $u \in \mathbb{R}^n$, then $\text{in}_w I \subseteq k[T^n]$ is homogeneous with respect to the grading given by $\deg x_i = u_i$.
- Definition: a *tropical basis* of $I \subseteq K[T^n]$ is a finite subset $B \subseteq I$ such that B generates I and such that

$$\text{trop}(V(I)) = \bigcap_{f \in B} \text{trop}(V(f)).$$

- Theorem: every ideal $I \subseteq K[T^n]$ has a tropical basis. [Set $J := I_{\text{proj}}$. Consider the finite Gröbner complex $\Sigma(J)$. For each cell P such that $J' := \text{in}_w J$ contains a monomial x^α for w in the relative interior of P, choose a $u \in \mathbb{R}^{n+1}$ such that $\text{in}_u J'$ is monomial. Then there is a unique monomial $g \in I$ that contains the term x^α and that otherwise has monomials outside $\text{in}_u J'$. For w in the relative interior of P we claim that $\text{in}_w g = x^\alpha$. Indeed, $h := \text{in}_w g - x^\alpha \in \text{in}_w J = J'$ and if h were non-zero, then $\text{in}_u h$ would be a linear combination of monomials *not* in $\text{in}_u J'$, a contradiction.

  Now the monomial $f := g(1, x_1, \ldots, x_n)$ lies in I. Doing this for all P whose relative interior needs to be cut out to get $\text{trop}(V(I))$ we get a finite set as in the theorem. Adding finitely many generators of I gives a tropical basis.
- Please read the rest of Section 2.6 by yourself!