TROPICAL GEOMETRY, LECTURE 7

JAN DRAISMA

1. Rest of §2.4

- Same assumptions as last time: $v : K^* \to \mathbb{R}$ surjective and with a section $w \mapsto t^w$.
- $\operatorname{Spec} R$ has two points, corresponding to the prime ideals $\{0\}$ and $\mathfrak{m}$. Since any proper ideal is contained in $\mathfrak{m}$, every non-empty closed set in the Zariski topology on $\operatorname{Spec} R$ contains $\mathfrak{m}$. Hence the closed sets are $\emptyset$, all of $\operatorname{Spec} R$ and the point corresponding to $\mathfrak{m}$. The point corresponding to $\{0\}$ is called the generic point, and the point corresponding to $\mathfrak{m}$ the closed point of $\operatorname{Spec} R$.
- $I \subseteq K[x_0, \ldots, x_n]$ homogeneous, $w \in \mathbb{R}^{n+1}$, then consider the ideal

$$I_R := \{t^{-\operatorname{trop}(f)(w)} f(t^{w_0} x_0, \ldots, t^{w_n} x_n) \mid f \in I\} \subseteq R[x_0, \ldots, x_n].$$

- Since $M = R[x_0, \ldots, x_n]/I_R$ is an R-algebra, we have a map $\operatorname{Spec} M \to \operatorname{Spec} R$. The fibre over the closed point $\mathfrak{m}$, called the *special fibre* is the set of prime ideals of M that intersect R in $\mathfrak{m}$, or the set of prime ideals in $R[x_0, \ldots, x_n]$ that contain $\mathfrak{m}$ and I_R. This is the same thing as prime ideals of $k[x_0, \ldots, x_n]$ containing the image of I_R, which is $\operatorname{in}_w I$. So the special fibre is isomorphic to $\operatorname{spec} k[x_0, \ldots, x_n]/\operatorname{in}_w I$.
- The fibre over the generic point, called the *general fibre* in the book, is the set of prime ideals of M intersecting R trivially. This is just the set of prime ideals P in $R[x_0, \ldots, x_n]$ containing I_R such that $P \cap R = \{0\}$. The set $Q := K \cdot P$ of all K-scalar multiples of elements of P is then a prime ideal in $K[x_0, \ldots, x_n]$ containing I_R, and you can go back via $P = Q \cap R[x_0, \ldots, x_n]$. Thus the general fibre is in bijection with $\operatorname{Spec}(K[x_0, \ldots, x_n]/K \cdot I_R) \cong \operatorname{Spec}(K[x_0, \ldots, x_n]/I)$ where the latter isomorphism is given by the map $K[x_0, \ldots, x_n]/I \to K[x_0, \ldots, x_n]/(K \cdot I_R)$ that sends x_i to $t^{w_i} x_i$.
- There is more in the book: M is a flat R-module with $M \otimes_R K \cong (K[x_0, \ldots, x_n]/I)$ and $M \otimes_R k \cong k[x_0, \ldots, x_n]/\operatorname{in}_w I$.
- Remark: we had chosen a splitting $\phi : \mathbb{R} \to K^*$ of the valuation to define $\operatorname{in}_w I$. Suppose we had chosen another splitting $\psi : \mathbb{R} \to K^*$, which gives rise to $\operatorname{in}'_w I$. Then $\operatorname{in}_w I$ is the image in $k[x_0, \ldots, x_n]$ of

$$\{\phi(-\operatorname{trop}(f)(w))(\phi(w_0)x_0, \ldots, \phi(w_n)x_n) \mid f \in I\}$$

  and $\operatorname{in}'_w I$ the image of

$$\{\psi(-\operatorname{trop}(f)(w))(\psi(w_0)x_0, \ldots, \psi(w_n)x_n) \mid f \in I\}.$$

  So $\operatorname{in}'_w I$ is the image of $\operatorname{in}_w I$ under the map $x_i \mapsto \overline{\psi(w_i)\phi(w_i)^{-1}} x_i$, which is an automorphism of $k[x_0, \ldots, x_n]$. So all algebraic invariants of these two ideals coincide.

2. From §2.5

- Given a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$, we'll define a polyhedral complex on $\mathbb{R}^{n+1}$, related to $\mathrm{trop}(V(I))$.
- For $w \in \mathbb{R}^{n+1}$ define
$$C_I[w] := \{w' \in \mathbb{R}^{n+1} \mid \mathrm{in}_{w'} I = \mathrm{in}_w I\} \subseteq \mathbb{R}^{n+1}$$
and let $\overline{C_I[w]}$ be its closure in the Euclidean topology.
- Example. Take $I = \langle (t + t^2)x_0^2 + x_1x_2 - tx_2^2 \rangle$ where $t = t^1 \in K$ is the image of $1 \in \mathbb{R}$ under the section, and take $w = (0, 1/2, 1/2)$. Then $\mathrm{in}_w I = \langle \overline{t^{-1}(t + t^2)x_0^2 + x_1x_2} \rangle = \langle x_0^2 + x_1x_2 \rangle$ and
$$\begin{aligned}\overline{C_I[w]} &= \overline{\{u \in \mathbb{R}^3 \mid 1 + 2u_0 = u_1 + u_2 < 1 + 2u_2\}} \\ &= \{u \in \mathbb{R}^3 \mid 1 + 2u_0 = u_1 + u_2 \le 1 + 2u_2\}.\end{aligned}$$
This is a half-plane with affine span the plane with equation $1 + 2u_0 = u_1 + u_2$. It is closed under adding $\mathbf{1} = (1, 1, 1)$. In $R^3/\mathbb{R}\mathbf{1}$ its image is a single ray.
- Proposition. The set $\overline{C_I[w]}$ is an $\mathbb{R}$-rational polyhedron whose lineality space contains $\mathbb{R}\mathbf{1}$. If $\mathrm{in}_w I$ is not monomial, then there exists a w' such that $\mathrm{in}_{w'} I$ is monomial and $\overline{C_I[w]}$ is a proper face of $\overline{C_I[w']}$.
- Pick u such that $\mathrm{in}_u \mathrm{in}_w I$ is monomial. By last week's work, $\mathrm{in}_{w+\epsilon u} I = \mathrm{in}_u \mathrm{in}_w I$ for $\epsilon > 0$ sufficiently small. Take $w' := w + \epsilon u$ for such an ϵ.

  Write $\mathrm{in}_{w'} I = \langle x^{\alpha_1}, \ldots, x^{\alpha_s} \rangle$. Also by last week, monomials not in $\mathrm{in}_{w'} I$ form a basis of S_K/I. Let $g_i' \in S_K$ be the linear combination of those monomials that satisfies $x^{\alpha_i} = g_i' \mod I$, so that $g_i := x^{\alpha_i} - g_i' \in I$ and no term of g_i' lies in $\mathrm{in}_{w'} I$. Then we must have $\mathrm{in}_{w'} g_i' = x^{\alpha_i}$.

  We have the following finite description of $C_I[w']$:
$$C_I[w'] = \{z \in \mathbb{R}^{n+1} \mid \forall i = 1, \ldots, s : \mathrm{in}_z g_i = x^{\alpha_i}\}.$$
Indeed, $\subseteq$ is clear because if $z \in C_I[w']$ then $\mathrm{in}_z g_i$ is a linear combination of monomials from $\mathrm{in}_{w'} I$, and the only such monomial in g_i is x^{α_i}. Conversely, if z lies in the RHS, then $\mathrm{in}_z I \supseteq \langle x^{\alpha_1}, \ldots, x^{\alpha_s} \rangle = \mathrm{in}_{w'} I$ but the two ideals have the same Hilbert function, hence they are equal.

  Note that the closure $\overline{C_I[w']}$ is now just the set of z for which x^{α_i} appears in $\mathrm{in}_z g_i$ for each i. This is clearly an $\mathbb{R}$-rational polyhedron. We claim that it contains $C_I[w]$. Indeed, if $\mathrm{in}_z I = \mathrm{in}_w I$, then $\mathrm{in}_u \mathrm{in}_z I = \mathrm{in}_u \mathrm{in}_w I = \mathrm{in}_{w'} I$ is a monomial ideal, and the only monomial of g_i in it is x^{α_i}. Thus $\mathrm{in}_u \mathrm{in}_z g_i$ must equal x^{α_i}, and hence $\mathrm{in}_z g_i$ must contain x^{α_i}.

  To prove that $\overline{C_I[w]}$ is also an $\mathbb{R}$-rational polyhedron we show that it is a face of $\overline{C_I[w']}$. For this, we argue that
$$C_I[w] = \{z \in \overline{C_I[w']} \mid \forall i : \mathrm{in}_z g_i = \mathrm{in}_w g_i\}.$$
For the inclusion $\subseteq$ consider z in the LHS. For each i, we have $\mathrm{in}_z g_i, \mathrm{in}_w g_i \in \mathrm{in}_w I$ and hence also $r := \mathrm{in}_z g_i - \mathrm{in}_w g_i \in \mathrm{in}_w I$. But this is a polynomial none of whose terms lies in $\mathrm{in}_u \mathrm{in}_w I = \mathrm{in}_{w'} I$, which is a contradiction unless $r = 0$. For the opposite inclusion, pick z in the RHS. Then clearly the ideal generated by the $\mathrm{in}_z g_i$'s contains the ideal generated by the $\mathrm{in}_w g_i$'s, which is already all of $\mathrm{in}_w I$ (argue via the Hilbert function of $\mathrm{in}_u \mathrm{in}_w I$). Hence $\mathrm{in}_z I$ contains $\mathrm{in}_w I$, hence they're equal, again by the Hilbert function.

The closure of $C_I[w]$ is now clearly a face of that of $C_I[w']$: it is the set of weight vectors that "pick out from g_i at least the terms that w picks out".

The lineality space statement follows from homogeneity.

- From now on, may think $w \in \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$. (Recall that $\mathrm{in}_w f = \mathrm{in}_{w+c\mathbf{1}} f$ if f is homogeneous).
- Theorem: the sets $\overline{C_I[w]}$, as w varies, form an $\mathbb{R}$-rational polyhedral complex with support all of $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.
- (In fact, a regular subdivision of the ambient space.)
- Lemma: $I \subseteq K[x_0, \ldots, x_n]$ homogeneous. Then the set of *monomial* ideals of the form $\mathrm{in}_w I$ is finite. (By the well-quasi-orderedness of monomial ideals, otherwise there would be w, w' with $\mathrm{in}_w I \subsetneq \mathrm{in}_{w'} I$ and both monomial. But both sides have the same Hilbert function, a contradiction.)
- Def: for a tropical polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ let Σ_F be the coarsest polyhedral complex such that F is affine-linear on each cell of Σ_F.
- Lemma: Let $A \in K^{r \times s}$ be of rank r and let $w \in \mathbb{R}^s$. Then we can permute the columns of A and the entries of w with the same permutation so as to achieve that the left-most $r \times r$-submatrix U of A is invertible, and such that the matrix $B := U^{-1}A$ has $v(b_{ij}) + w_j \geq w_i$ for all i, j.
- Replace A by the matrix $A \cdot \mathrm{diag}(t^{w_1}, \ldots, t^{w_s})$, and permute the columns of this new A such that the minimum of $v(\det(A[J]))$ over all r-subsets $J \subseteq \{1, \ldots, s\}$ is attained by $J = \{1, \ldots, r\}$. Let U and B be as in the lemma. Then we know that $v(\det B[J]) \geq 0$ for all r-sets J, with equality for $J = \{1, \ldots, r\}$. What we need to prove boils down to $v(b_{ij}) \geq 0$ for all i, j. This is clearly true for $j \leq r$ (where $b_{ij} = \delta_{ij}$). So suppose, w.l.o.g., that $v(b_{i,r+1}) < 0$. Then set $J := \{1, \ldots, i-1, i+1, \ldots, r, r+1\}$ satisfies $v(\det(B[J])) = v(b_{i,r+1}) < 0$, a contradiction.

3. Homework

From §2.7, do exercises 16 and 26.

Furthermore, consider the ideal $I \subseteq K[x, y, z, u]$ generated by the two quadrics $f = x^2 + y^2 + z^2 + u^2$ and $h = xy + yz + zu + ux$.

(1) Determine the Hilbert function of I.
(2) Determine all monomial initial ideals $\mathrm{in}_w I$.
(3) Determine the polynomial g from (2.5.2) in the book.
(4) Describe $\Sigma_{\mathrm{trop}(g)}$.