TROPICAL GEOMETRY, LECTURE 6

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- Same assumptions as last time: $v : K^* \to \mathbb{R}$ surjective and with a section $w \mapsto t^w$.
- Lemma: let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal and let $w, u \in \mathbb{R}^{n+1}$. Then for sufficiently small $\epsilon > 0$ we have $\mathrm{in}_u \mathrm{in}_w I \subseteq \mathrm{in}_{w+\epsilon u} I$.
- Indeed, the RHS is generated by finitely many elements $\mathrm{in}_u \mathrm{in}_w f$ with $f \in I$, and for ϵ sufficiently small these are equal to $\mathrm{in}_{w+\epsilon u} f$.
- Lemma: Let $J \subseteq k[x_0, \ldots, x_n]$ be a homogeneous ideal. Then there exists an open set $U \subseteq \mathbb{R}^{n+1}$ such that for $u \in U$ the ideal $\mathrm{in}_u J$ is constant and generated (hence spanned) by monomials.
- Indeed, let $M_u \subseteq \mathrm{in}_u J$ be the ideal generated by all monomials in $\mathrm{in}_u J$. By Noetherianity of $k[x_0, \ldots, x_n]$ there exists a u such that M_u is not strictly contained in any $M_{u'}$. We claim that $M_u = \mathrm{in}_u J$. Indeed, let $f_1, \ldots, f_s \in J$ such that the $m_i := \mathrm{in}_u f_i$ are monomials generating M_u, and pick $f \in J$. We need to show that all terms in $\mathrm{in}_u f$ lie in M_u. Pick a term m corresponding to a vertex of the Newton polytope of f, and choose a linear function $u' \in \mathbb{R}^{n+1}$ for which that vertex is the minimiser. Then for ϵ sufficiently small, $\mathrm{in}_{u+\epsilon u'} f$ is m, while $\mathrm{in}_{u+\epsilon u'} f_i$ remains m_i. Thus $M_{u+\epsilon u'} \supseteq M_u$, and by maximality of M_u equality holds. Thus m lies in M_u, say it is a multiple tm_i with t a single term. Then $f - tf_i \in J$ and $\mathrm{in}_u(f - tf_i) = \mathrm{in}_u f - m$, and by induction we on the number of terms of $\mathrm{in}_u f$ we're done. Finally, for each u' in some open neighbourhood of u we have $\mathrm{in}_{u'} f_i = m_i$, hence $M_{u'} = M_u$ by maximality of M_u, and hence $\mathrm{in}_{u'} J = M_{u'} = M_u$ by the argument above applied to u' instead of u.
- Lemma: let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal, and fix $w \in \mathbb{R}^{n+1}$. Then there exists a $u \in \mathbb{R}^{n+1}$ and an $\epsilon > 0$ such that both ideals $\mathrm{in}_u(\mathrm{in}_w(I))$ and $\mathrm{in}_{w+\epsilon u}$ are generated (hence spanned) by monomials, and the first is contained in the latter.
- By the previous lemma applied to $J := \mathrm{in}_w(I)$, we can choose an open $U \subseteq \mathbb{R}^{n+1}$ such that $\mathrm{in}_u(\mathrm{in}_w(I))$ is monomial and constant for $u \in U$, say generated by the monomials $m_i := \mathrm{in}_u(\mathrm{in}_w f_i)$ with $f_i \in I$ for $i = 1, \ldots, s$. We can take U so small and $\delta > 0$ such that, in fact, $m_i = \mathrm{in}_{w+\epsilon u} f_i$ for all $u \in U$ and $\epsilon \in (0, \delta)$. For these (u, ϵ) we then have $\mathrm{in}_u \mathrm{in}_w I \subseteq \mathrm{in}_{w+\epsilon u} I$.

  Among these pairs, choose a pair for which the ideal generated by the monomials in $\mathrm{in}_{w+\epsilon u} I$ is maximal. By an argument like above, using that we can move u by a small amount without changing $\mathrm{in}_u(\mathrm{in}_w(I))$, we find that $\mathrm{in}_{w+\epsilon u} I$ is generated by monomials.
- Recall Hilbert functions of homogeneous ideals $I \subseteq S_K := K[x_0, \ldots, x_n]$ and $J \subseteq S_k := k[x_0, \ldots, x_n]$: they are the maps $d \mapsto \dim_K (S_K)_d/(I)_d$, where $(.)_d$ stands for those polynomials of degree exactly d. (Note: in the commutative algebra course, we had *at most* d.) For $d \gg 0$ this is

a polynomial in d of degree one less than the Krull dimension of S_K/I. Similarly for J.

- Lemma: Let $I \subseteq S_K$ be homogeneous, $d \in \mathbb{Z}_{\geq 0}$, and $w \in \mathbb{R}^{n+1}$ such that $\mathrm{in}_w(I)_d$ is spanned over k by its monomials. Then the monomials of degree d that are not in $\mathrm{in}_w(I)$ form a K-basis for $(S_K/I)_d$.
- Indeed, suppose that $f := \sum_{i=1}^{s} c_i m_i \in I_d$, where the m_i are distinct monomials of degree d not in $\mathrm{in}_w(I)$, the $c_i \in K$ are nonzero coefficients, and $s > 0$. Then the monomials appearing in the nonzero polynomial $\mathrm{in}_w f$ are all in $\mathrm{in}_w(I)$ but also not in $\mathrm{in}_w(I)$, a contradiction. This shows that the monomials in the lemma are linearly indepencent. In particular, we have $\dim_K(S_K/I)_d \geq \dim_k(S_k/\mathrm{in}_w(I))_d$.

  For the converse inequality, let the monomials $m_1, \ldots, m_s$ span $\mathrm{in}_w I_d$ and choose $f_i \in I_d$ with $\mathrm{in}_w f_i = m_i$. Then $f_1, \ldots, f_s$ are linearly independent in S_K. Indeed, if $f := \sum_i a_i f_i \in I$ with not all coefficients a_i zero, then $\mathrm{in}_w f$ is the sum of the $\mathrm{in}_w(a_i f_i)$ with $v(a_i) + \mathrm{trop}(f_i)(w)$ minimal, since these do not cancel. But this is a nonzero linear combination of monomials, hence in particular f is nonzero.
- Corollary: for any $w \in \mathbb{R}^{n+1}$ and any homogeneous ideal $I \subseteq S_K$, the Hilbert function of I agrees with that of $\mathrm{in}_w(I) \subseteq S_k$.
- Proof: pick $u \in \mathbb{R}^{n+1}$ and $\epsilon > 0$ such that $J_1 := \mathrm{in}_u(\mathrm{in}_w(I)) \subseteq J_2 := \mathrm{in}_{w+\epsilon u} I$ and both are monomial ideals. By the previous lemma, monomials not in J_2 form a basis of S_K/I, and monomials not in J_1 form a basis of $S_k/\mathrm{in}_w(I)$. Thus it remains to show that $J_1 = J_2$. Let m be a monomial in $J_2 \setminus J_1$. Thus $m = \mathrm{in}_{w+\epsilon u} f$ for some $f \in I$ such that no monomial in $\mathrm{in}_w f$ is in J_1. This means that $\mathrm{in}_w f$ is not in $\mathrm{in}_w(I)$, a contradiction.
- Remark: in particular, the dimension of the variety $\subseteq \mathbb{P}^n_k$ defined by $\mathrm{in}_w I$ equals that of the variety $\subseteq \mathbb{P}^n_K$ defined by I.
- Corollary: let $I \subseteq S_K$ be a homogeneous ideal. Then for any $w, u \in \mathbb{R}^{n+1}$ we have $\mathrm{in}_u(\mathrm{in}_w I) = \mathrm{in}_{w+\epsilon u} I$ for all sufficiently small ϵ.
- The inclusion $\subseteq$ for sufficiently small ϵ is the first lemma. Since both ideals have the same Hilbert function, equality holds.
- In last week's example: $I = \langle x_0 + 2x_1 - 3x_2, 3x_1 - 4x_2 + 5x_3 \rangle$ over $\mathbb{Q}$ with the 2-adic valuation we had $\mathrm{in}_{(0,0,0,0)} I = \langle x_0 + x_2, x_1 + x_3 \rangle$ and $\mathrm{in}_{0,0,\epsilon,\epsilon} I = \langle x_0, x_1 \rangle$. The monomials of degree d not in this ideal are $x_2^d, x_2^{d-1} x_3, \ldots, x_3^d$, so the Hilbert function is $d \mapsto d + 1$. This is consistent with the fact that $V(I) \subseteq \mathbb{P}^3_Q$ is a projective line, as is $V(\mathrm{in}_{(0,0,0,0)} I) \subseteq \mathbb{P}^3_{\mathbb{F}_2}$.
- Proposition: a homogeneous Gröbner basis $G \subseteq I$ of a homogeneous ideal $I \subseteq S_K$ with respect to a weight vector $w \in \mathbb{R}^{n+1}$ generates I as an ideal.
- Let $I' \subseteq I$ be the homogeneous ideal generated by G. Then $\mathrm{in}_w I' \subseteq \mathrm{in}_w I$, but the LHS contains the generators $\mathrm{in}_w g,\ g \in G$ of the RHS, so equality holds. By the corollary above, $\dim_K(S_K/I')_d = \dim_k(S_k/\mathrm{in}_w I')_d = \dim_k(S_k/\mathrm{in}_w I) = \dim_K(S_K/I)_d$. Together with the inclusion $I' \subseteq I$ this implies that $I' = I$.
- Proposition: let $w \in \Gamma^{n+1}$ and assume that $I \subseteq S_K$ is a homogeneous prime ideal such that S_K/I has Krull dimension d. Then for each minimal prime J of $S_k/\mathrm{in}_w I$ the ring S_k/J has Krull dimension d.
- Example: let $I = \langle xy + z^2 \rangle \subseteq K[x, y, z]$ and $w = (0, 0, 1)$. Then $\mathrm{in}_w I = \langle xy \rangle \subseteq k[x, y, z]$ and the minimal primes above it are $\langle x \rangle$ and $\langle y \rangle$.

- Proposition: Let $I \subseteq K[x_0, \ldots, x_n]$ be an ideal generated by linear forms, and let $w \in \mathbb{R}^{n+1}$ such that $w \in \text{Trop}(V_{T^{n+1}}(f))$ for every circuit f. Then there is a point $p \in V_{T^{n+1}}(I)$ such that $v(p) = w$.
- As usual, reduce to the case $w = 0$. Let d be the dimension of the linear space in $\mathbb{A}_K^{n+1}$ defined by I. By the above, d is also the dimension of the subvariety of $\mathbb{A}_k^{n+1}$ defined by $\text{in}_w I$. This variety is contained in $V(J)$, where $J \subseteq \text{in}_w I$ is the ideal generated by $(\text{in}_w I)_1$. Every set $S \subseteq \{1, \ldots, n\}$ of variables that are linearly dependent on $V(I)$ are also linearly dependent on $V(J)$, since the initial form of a linear form with support contained in S has support contained in S. Hence $V(J)$ has dimension equal to d (And, since $V(J)$ is irreducible, $V(\text{in}_w(I)) = V(J)$. In fact, equality of ideals holds.) Now pick a basis $B \subseteq \{0, \ldots, n\}$ such that the $x_i, i \in B$ form a basis of $(S_k/J)_1$. This is then also a basis for $(S_K/I)_1$.

  We claim that J does not contain any single variables. Suppose that, on the contrary, $f = \sum_j a_j x_j \in I$ with $\text{in}_0 f = x_i$, which means that the coefficients of $x_j,\ j \neq i$ all have valuations > 0. Moreover, we may assume that f has minimal support among linear forms with this property. If f is a circuit, then this contradicts $0 \in \text{trop}(V(f))$. Otherwise, there is a circuit $f' = \sum_j b_j x_j$ whose support is contained in that of f and does not contain the index i. Consider, for $c \in K$, the combination $f - cf' \in I$. The coefficient of x_j is $a_j - cb_j$, of which the valuation is at least the minimum of $v(a_j)$ and $v(c) + v(b_j)$. We want this to be strictly positive, which is guaranteed by $v(c) \geq v(a_j/b_j)$. So we pick $c = a_j/b_j$ for the j with $v(a_j/b_j)$ minimal. Then we find that $f - f'$ is a linear form with smaller support but still with $\text{in}_0(f - f') = x_i$.

  OK, so J contains no single variable. Hence $V(J)$ contains a point $q \in (k^*)^{n+1}$. Now pick $p_j, j \in B$ arbitrary lifts of the $q_j, i \in B$. Then there is a unique point $p \in V_{\mathbb{A}^{n+1}}(I)$ with these d coordinates. We claim that it lies in R^{n+1} and that in fact $\overline{p_i} = q_i$ also for $i \notin B$. Indeed, for such i there is a unique circuit $f = \sum_{j \in B+i} a_j x_j \in I$, say scaled such that the minimal valuation among the a_j is 0. Then $\text{in}_0 f$ has support contained in $B + i$ and containing i since B is a basis for $\text{in}_0 J$, as well. This means that the coefficient a_i has valuation 0, and reducing mod $\mathfrak{m}$ we find that $\overline{p_j}, j \in B+i$ satisfy the equation $\sum_{j \in B} \overline{a_j} q_j + \overline{a_i p_i} = 0$. Hence $\overline{p_i} = q_i$ with valuation 0 as required.
- Interesting special case for later: let A be the linear map $K^m \to K^{\binom{m}{2}}$, where the coordinates on the left-hand side are called y_i and those on the right-hand side x_{ij} with $i < j$, defined by $x_{ij} = y_i - y_j$. Let $V \subseteq T^{\binom{m}{2}}$ be the image intersected with the torus.
- What is the dimension of the image?
- What are the circuits for V? This is where the name "circuits" comes from.
- What are subsets of $\{1, \ldots, \binom{m}{2}\}$ corresponding to bases? (The spanning trees!)
- Show that for each basis there is a maximal cone in $\text{trop}(V)$.