TROPICAL GEOMETRY, LECTURE 5

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1. Tropicalised linear spaces

- Preliminary remark: for any subvariety $X \subseteq T^n$ and any $p \in X$ it is clear that $v(p) \in \text{trop}(X)$. (Later we will see a converse to this—the *Fundamental Theorem*.)
- Preliminary remark: today, whenever convenient, we will assume that $v(K^*) = \mathbb{R}$, i.e., that the value group is all of $\mathbb{R}$, and that we have a section $\mathbb{R} \to K^*,\ w \mapsto t^w$. This can always be achieved by enlarging K.
- Example: Linear spaces. Let I be generated by linear forms in the variables x_i, and that it does not contain variables, so that $V(I) \subseteq T^n$ is non-empty. The support of a linear form is the set $J \subseteq \{1, \ldots, n\}$ of indices of variables appearing in it. Let S be the collection of supports of linear forms in I. For each inclusion-minimal nonempty support in S there is, up to scaling, a unique linear form in I with that support. Let C be the finite set of such representatives, one for each inclusion-minimal support; these are the *circuits* of the linear space.
- Theorem: if I is generated by linear forms, then $\text{trop}(V(I)) = \bigcap_{f \in C} \text{trop}(V(f))$.
- The inclusion $\subseteq$ is immediate. We'll prove the converse in the special case where I is generated by linear forms with coefficients in a subfield L of K on which the valuation is trivial. (This is the *constant coefficient case*, think $L = \mathbb{C}$ and K is a field of suitable formal power series. The general case is proposition 4.1.6 in the book, which we may discuss later.) Then we may take the elements of C with coefficients from L.

  So pick a point w on the right. We will construct a point p in $V(I)$ (over the larger field K) with $v(p) = w$, so that $w \in \text{trop}(V(I))$ by the preliminary remark above.
- Call a subset of $\{1, \ldots, n\}$ a *basis* if it does not contain an element of S, i.e., if the corresponding variables are linearly independent on $V(I)$, and if it is moreover maximal with this property. By linear algebra, all bases have the same cardinality, namely, the dimension d of $V(I)$. For each basis B and $i \notin B$ there is a unique circuit with support contained in $B + i$: $f_{B+i} = \sum_{j \in B+i} a_{B,i,j} x_j$ in C. This has $a_{B,i,i} \neq 0$; and for each j with $a_{B,i,j} \neq 0$ also $B - j + i$ is a basis. Choose B such that $\sum_{j \in B} w_j$ is *maximal* among all bases, and for each $j \in B$ choose $p_j \in K$ sufficiently arbitrary with $v(p_j) = w_j$. For $i \notin B$ set $p_i := -(\sum_{j \in B} a_{B,i,j})/a_{B,i,i}$. Then by construction $p \in V(I)$. I claim that $v(p_i) = w_i$ for $i \notin B$, as well. Indeed, we have $w_i \leq \min_{j \in B: a_{B,i,j} \neq 0} w_j$, or else we could pick a j attaining the minimum and then $B - \{j\} + \{i\}$ would be a basis with a larger sum of weights. Moreover, since $f \in I$, we have $w \in \text{trop}(V(f))$, and hence w_i cannot be strictly smaller than $\min_{j \in B} w_j$. This means that we expect p_i

to have valuation $\min_{j\in B} w_j = w_i$, and "sufficiently arbitrary" ensures that this is indeed the valuation of p_i.

2. §2.4, START

- Will work with *homogeneous ideals* $I \subseteq K[x_0,\ldots,x_n]$.
- Definition: for nonzero $f = \sum_\alpha c_\alpha x^\alpha \in K[\mathbb{P}^n]$ and $w \in \mathbb{R}^{n+1}$, set $W := \mathrm{trop}(f)(w)$. The *initial form* of f is $\mathrm{in}_w f = \sum_{\alpha:v(c_\alpha)+w\cdot\alpha=W} \overline{c_\alpha t^{-v(c_\alpha)}} x^\alpha \in k[\mathbb{P}^n]$.
- Alternative characterisation: a term $c_\alpha x^\alpha$ in f gives a term $t^{-W} c_\alpha t^{w\cdot\alpha} x^\alpha$ in $g := t^{-W} f(t^{w_0}x_0,\ldots,t^{w_n}x_n)$. Hence this latter polynomial has all coefficients of valuation ≥ 0, by the choice of W. Then we have $\mathrm{in}_w f = \overline{g} \in k[\mathbb{P}^n]$. Indeed, the terms that survive on the right are those with $v(c_\alpha)+w\cdot\alpha = W$, so that $t^{-v(c_\alpha)} = t^{-W} t^{w\cdot\alpha}$.
- Lemma: if $\mathrm{trop}(f)(w) < \mathrm{trop}(g)(w)$, then $\mathrm{in}_w(f+g) = \mathrm{in}_w(f)$.
- Lemma: consider nonzero $f, g \in K[\mathbb{P}^n]$, with $\mathrm{trop}(f)(w) =: W$ and $\mathrm{trop}(g)(w) =: W$. Then $h := f + t^{W-V} g$ has initial form $\mathrm{in}_w(f) + \mathrm{in}_w(g)$ unless this is zero, in which case $\mathrm{trop}(h)(w) > \mathrm{trop}(f)(w)$.

  (Indeed, all terms ex^γ of h satisfy $v(e) + w\cdot\gamma \geq W$. Suppose that equality holds for at least some term. Then we have
  $$\begin{aligned}\mathrm{in}_w(h) &= \overline{t^{-W} h(t^{w_1}x_1,\ldots,t^{w_n}x_n)} \\ &= \overline{t^{-W} f(\ldots)} + \overline{t^{-V} g(\ldots)} \qquad = \mathrm{in}_w(f) + \mathrm{in}_w(g).\end{aligned}$$
)
- Lemma: $\mathrm{in}_w(fg) = \mathrm{in}_w(f)\mathrm{in}_w(g)$. Indeed, if $W = \mathrm{trop}(f)(w)$ and $V = \mathrm{trop}(g)(w)$, then $\mathrm{trop}(fg)(w) = W + V$ and the left-hand side equals
  $$\overline{t^{-W-V}(fg)(t^{w_1}x_1,\ldots,t^{w_n}x_n)} = \mathrm{in}_w(f)\mathrm{in}_w(g).$$
  A particular instance that we might use is the case where f is a single term cx^α. Then $\mathrm{in}_w(fg) = \overline{t^{-v(c)}c} x^\alpha \mathrm{in}_w(g)$.
- Lemma: if f is homogeneous, then $\mathrm{in}_w f = \mathrm{in}_{w+c\mathbf{1}} f$ for all $c \in \mathbb{R}$, where $\mathbf{1} = (1,\ldots,1)$. (N.B. this implies that the tropical variety of a homogeneous ideal is invariant under translation by the vector $(1,\ldots,1)$.)
- Definition: $\mathrm{in}_w I := \langle\{\mathrm{in}_w f \mid f \in I \setminus \{0\}\}\rangle \subseteq k[x_0,\ldots,x_n]$.
- Lemma: this is a homogeneous ideal in $k[x_0,\ldots,x_n]$, and it equals $\{\mathrm{in}_w(f) \mid f \in I \setminus \{0\}\} \cup \{0\}$.

  (Homogeneity follows from the fact that I is homogeneous and the initial form of a homogeneous form is homogeneous. For the last statement, consider nonzero linear combination $h = \sum a_i \mathrm{in}_w(f_i)$ where the $a_i \in k[x_0,\ldots,x_n]$ and $f_i \in I$. Find $q_i \in K[x_0,\ldots,x_n]$ with $\mathrm{in}_w(q_i) = a_i$ and scaled such that all $\mathrm{trop}(a_i f_i)(w)$ are equal. Then $h = \mathrm{in}_w(\sum_i q_i f_i)$.)
- Definition. A *Gröbner basis* of I w.r.t. w is a set $\{g_1,\ldots,g_s\} \subseteq I$ such that $\mathrm{in}_w(I) = \langle \mathrm{in}_w g_1,\ldots,\mathrm{in}_w g_s\rangle$. By Noetherianity of $k[x_0,\ldots,x_n]$, any homogeneous ideal I has a Gröbner basis w.r.t. any weight vector w.
- Proposition: a homogeneous Gröbner basis of I w.r.t. w generates I. There is an easy reduction to the case where $w = 0$. Then the statement is: if $g_1,\ldots,g_s \in I \cap R[x_0,\ldots,x_n] =: J$ are homogeneous and such that $\overline{g_1},\ldots,\overline{g_s}$ generate the image of J in $k[x_0,\ldots,x_n]$, then $g_1,\ldots,g_s$ generate I. The proof is somewhat tricky, and we skip it for now.

- Example: $K = \mathbb{Q}$ with the 2-adic valuation and $I = \langle x_0 + 2x_1 - 3x_2, 3x_1 - 4x_2 + 5x_3 \rangle$. For $w = (0, 0, 0, 0)$ the initial forms are $x_0 + x_2, x_1 + x_3 \in \mathbb{F}_2[x]$, respectively, and these turn out to generate $\text{in}_0 I$. Indeed, let f be any element in I, w.l.o.g. with $\text{trop}(f)(0) = 0$. Replacing in f each x_0 by $-2x_1 + 3x_2$ and each x_1 by $(4/3)x_2 - (5/3)x_3$ yields zero. Note that all these coefficients have nonnegative valuation. But this means that replacing in $\text{in}_0 f$ the x_0 by x_2 and the x_1 by x_3 must also yield zero.

  Let's do another one: $w = (1, 0, 0, 1)$. We have $\text{in}_w I = \text{in}_0 \tilde{I}$ where $\tilde{I}$ arises from I by replacing each x_i by $t^{w_i} x_i$. In this case, we can take $t = 2$, so $\tilde{I}$ equals $\langle 2x_0 + 2x_1 - 3x_2, 3x_1 - 4x_2 + 10x_3 \rangle$, with initial ideal $\langle x_2, x_1 \rangle$.
- Remark: if $K \to k$ admits a section $k \to K$, and if I is a homogeneous ideal generated by elements from $k[x_0, \ldots, x_n]$, then for w consisting of positive reals linearly independent over $\mathbb{Q}$, we obtain a monomial order by declaring that $x^\alpha > x^\beta$ if $w \cdot \alpha > w \cdot \beta$, and an ordinary Gröbner basis for I w.r.t. this monomial order with coefficients in k is a GB in the sense w.r.t. $-w$.
- For a polynomial $h \in k[x_0, \ldots, x_n]$ and $w \in \mathbb{R}^{n+1}$ we also write $\text{in}_w h$ for the sum of the terms $c_\alpha x^\alpha$ with $w \cdot \alpha$ minimal. This is consistent with the notation above, if we think of k as a field with trivial valuation.
- Lemma: let $w', w \in \mathbb{R}^{n+1}$ and $f \in K[x_0, \ldots, x_n]$. Then for sufficiently small $\epsilon > 0$ we have $\text{in}_{w'}(\text{in}_w f) = \text{in}_{w+\epsilon w'} f$.

  (For ϵ sufficiently small, a term $c_\alpha x^\alpha$ of f contributes to the LHS only if $v(c_\alpha) + w \cdot \alpha$ is minimal, namely equal to $W = \text{trop}(f)(w)$, and among these only those for which $v(c_\alpha) + (w + \epsilon w') \cdot \alpha = W + \epsilon(w' \cdot \alpha)$ is minimal, i.e., those for which $w' \cdot \alpha$ is minimal, say W'. Then it contributes $\overline{t^{-v(c_\alpha)} c_\alpha} x^\alpha$. This is the same as on the right.)