TROPICAL GEOMETRY, LECTURE 4

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1. MS §3.1 Tropical hypersurfaces

- Let (K, v) be a valued field with valuation ring R having maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$.
- For non-zero $f \in K[T^n], f = \sum_\alpha c_\alpha x^\alpha$ we define $\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R}^n, w \mapsto \min_\alpha(v(c_\alpha) + \alpha \cdot w)$, the *tropicalisation* of f.
- Definition: the tropical hypersurface defined by f is $\mathrm{Trop}(V(f)) := \{w \in \mathbb{R}^n \mid$ the minimum in $\mathrm{trop}(f)$ is achieved at least twice$\}$. Equivalently, this is the set where $\mathrm{trop}(f)$ is nondifferentiable (or, equivalently, nonlinear).
- Remark: This is a union of Γ-rational polyhedra, where Γ is the value group (in particular, a union of polyhedral cones if the valuation is trivial).
- Remark: suppose that $\Gamma \subseteq \mathbb{R}$ is divisible, i.e., a $\mathbb{Q}$-vector space. Then the set of Γ-valued points in any Γ-rational polyhedron is dense.
- Examples: tropical curves in the plane.
- Higher-dimensional example: the tropical determinant. Let K be arbitrary, $n = m^2$ with coordinates $x_{ij}, i, j = 1, \ldots, m$, $f = \det(x)$. $\mathrm{tdet} := \mathrm{trop}(\det)$ is the function that assigns to $w \in \mathbb{R}^{m\times m}$ the minimum of $\sum_i w_{i\pi(i)}$ over all permutations $\pi \in S_m$.

  Clearly, if $a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{R}$ such that $a_i + b_j \leq w_{ij}$ for all i, j, then $\mathrm{tdet}(w) \geq \sum_i a_i + \sum_j b_j$.

  Theorem (Egerváry, look up for yourself): for each $w \in \mathbb{R}^{3\times 3}$ there exist $a_1, \ldots, b_m \in \mathbb{R}$ such that equality holds.

  For these values, if π minimises the sum, then for each i, $w_{i\pi(i)}$ must be *equal* $a_i + b_{\pi(i)}$.

  Thus, given a collection S of permutations, we can parameterise the w for which those permutations are among the minimisers by choosing arbitrary numbers $a_1, \ldots, b_m$, setting $w_{ij} = a_i + b_j$ if $\pi(i) = j$ for some $\pi \in S$, and $w_{ij} \geq a_i + b_j$ if no such π exists.

  Consider the bipartite subgraph $\Sigma = \Sigma_S$ of $K_{m,m}$ with edges (i, j) if $\exists \pi \in S$ such that $\pi(i) = j$; thus the edges of Σ are the union of a number of perfect matchings. Adding to all a_i with i in a connected component C a real number t and subtracting t from the j in C yields the same w_{ij} for (i, j) an edge in C. Thus the set above is a polyhedral cone of dimension $2m$ minus the number of connected components of Σ_S (this counts the degrees of freedom for the w_{ij} appearing in minimisers) plus the number of non-edges of Γ (this counts the degrees of freedom for the remaining w_{ij}).

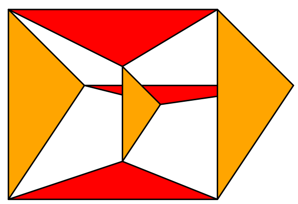
  Specialise to $m = 3$. Up to $S_3 \times S_3$, there are several types of Σ_S, namely:

  (1) Edges $(1, 1), (2, 2), (3, 3)$. This gives a $6 - 3 + 6 = 9$-dimensional cone of w's. There are 6 of these. These cones do *not* lie in $\mathrm{trop}(V(f))$.

(2) Edges $(1,1),(1,2),(2,2),(2,1),(3,3)$. This gives a $6-2+4=8$-dimensional cone. There are $3 \cdot 3 = 9$ of these.
(3) Edges $(1,1),(1,2),(2,2),(2,3),(3,3),(3,1)$. This gives a $6-1+3=8$-dimensional cone. There are 6 of these.
(4) Edges $(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)$. This gives a $6-1+2=7$-dimensional cone. There are $9 \cdot 2 = 18$ of these.
(5) All edges but one. This gives a $6-1+1=6$-dimensional cone. There are 9 of these.
(6) All edges. This gives a $6-1=5$-dimensional cone.

This last space is the intersection of the lineality spaces of all cones, which is the space of all sum matrices w, i.e., those with $w_{ij} = a_i + b_j$ for all i,j and suitable a,b.

We find that $\text{trop}(v(f))$ is a polyhedral fan of dimension 8 in 9-space. Modulo its lineality space it is a 3-dimensional fan in 4-space. Intersecting with a 3-sphere gives a *spherical complex* consisting of 6 triangles, 9 quadrangles, 18 edges, and 9 vertices. It looks like this:



- Kapranov's Theorem: Assume that K is algebraically closed with a non-trivial valuation. Then the following two sets are equal:
  (1) $\text{trop}(V(f)) \subseteq \mathbb{R}^n$; and
  (2) the closure in the Euclidean topology of $\{(v(p_1),\ldots,v(p_n)) \mid p \in V(f) \subseteq T^n\}$.

  (Note that the value group Γ is a $\mathbb{Q}$-vector space dense in $\mathbb{R}$; and that the theorem justifies the notation $\text{trop}(V(f))$ to some extent.)

  The inclusion $\supseteq$ is easy. The opposite inclusion we have already seen in the special case where $n=1$ (using Gauss's lemma). Since $\text{trop}(V(f))$ is the union of Γ-rational polyhedra, the set of Γ-valued points in $\text{Trop}(V(f))$ is dense in it. So we need only show that if $(w_1,\ldots,w_n) \in \text{Trop}(V(f)) \cap \Gamma^n$, then there exists a $p \in V(f)$ with $v(p_i) = w_i$.

  For convenience, choose a section $\Gamma \to K^*, w \mapsto t^w$ of the valuation map. Set $W := \text{trop}(f)(w)$. Then consider the polynomial

$$g := t^{-W} f(t^{w_1}x_1,\ldots,t^{w_n}x_n).$$

  A term $c_\alpha x^\alpha$ gives rise to a term $t^{-W}c_\alpha t^{\alpha \cdot w} x^\alpha$ in g, of which the valuation is $v(c_\alpha) + \alpha \cdot w - W \geq 0$. We will show that there is a point $q \in V(g)$ such that $v(q_i) = 0$ for all i; then the point $p := (t^{w_1}q_1,\ldots,t^{w_n}q_n)$ has valuation vector $(w_1,\ldots,w_n)$ and is in $V(f)$.

  Now consider the reduction $\overline{g} \in k[x_1,\ldots,x_n]$. This is a polynomial with at least two terms, since $w \in \text{Trop}(V(f))$. Hence there is a variable, say x_n, which appears with at least two distinct exponents in $\overline{g}$. Write $g = \sum_i g_i x_n^i$ with $g_i \in K[T^{n-1}]$; so there exists $d < e$ with $\overline{g_d}, \overline{g_e} \neq 0$.

Choose a point $\overline{q} \in (k^*)^{n-1}$ where $\overline{g_d}, \overline{g_e}$ are non-zero, and lift to a point $q \in R^{n-1}$. Hence $g_d(q), g_e(q)$ have valuation zero. Now consider the polynomial $h(y) := g(q_1, \ldots, q_{n-1}, y) \in K = [y]$. It satisfies:

(1) $\mathrm{trop}(h)(0) = 0$,

(2) $\overline{h}$ has at least two terms, and hence a root in k^*; lift this to a $r \in R$.

Then $h(r) \in \mathfrak{m}$ so $v(h(r)) > 0$ while $v(r) = 0$, hence $\mathrm{trop}(h)$ has a tropical root at 0. Now, by Gauss's Lemma, h itself has a root q_n with valuation 0.

- Remark: using a lemma from last time, the proof above shows that, for $w \in \mathrm{trop}(V(f)) \cap \Gamma^n$, the set of points $p \in V(f)$ with $v(p) = w$ is Zariski-dense in $V(f)$. Indeed, its projection into K^{n-1} contains a dense subset.
- A consequence of Kapranov's theorem is that $\mathrm{trop}(V(fg)) = \mathrm{trop}(V(f)) \cup \mathrm{trop}(V(g))$.
- Let P be the convex hull in $\mathbb{R}^{n+1}$ of the points $(\alpha, v(c_\alpha))$ for $c_\alpha \neq 0$. Let $w \in \mathbb{R}^n$ and let $F_w := \mathrm{face}_{(w,1)}(P)$. This is one of the lower faces of P, hence projects down to one of the faces in the corresponding regular subdivision of the Newton polytope of f. For each lower face F of P, the set

$$Q_F := \{w \in \mathbb{R}^n \mid F_w \supseteq F\}$$

is a Γ-rational polyhedron in $\mathbb{R}^n$. Some easily verifiable facts:

(1) If F' is a face of F in the subdivision, then Q_F is a face of $Q_{F'}$.

(2) The Q_F form a polyhedral complex, and the map $F \to Q_F$ is a bijection sending a d-dimensional face of the subdivision to an $(n-d)$-dimensional face.

(3) $\mathrm{trop}(V(f))$ is the union of the $(d-1)$-dimensional polyhedra Q_F.

- Example 3.1.9.

2. Tropical varieties

- For an ideal $I \subseteq K[T^n]$ and its corresponding variety $X = V(I) \subseteq T^n$ we define $\mathrm{trop}(X) = \bigcap_{f \in I} \mathrm{trop}(V(f))$, the *tropicalisation* of X or *tropical variety associated to* X. Strictly speaking, it depends on I and not just on X, but when K is algebraically closed, then it depends only on X, since, by the Nullstellensatz, $I_X = \sqrt{I}$ and the following lemma holds.
- Lemma: $\bigcap_{f \in I} \mathrm{trop}(V(f)) = \bigcap_{f \in \sqrt{I}} \mathrm{trop}(V(f))$. Indeed, the RHS is clearly contained in the LHS. For the opposite, note that if $f \in \sqrt{I}$, then $f^d \in I$ for some $d \geq 1$, and by the consequence to Kapranov's theorem, $\mathrm{trop}(V(f^d)) = \mathrm{trop}(V(f))$.
- Lemma: When $I = (f)$, the definition above agrees with our definition earlier. Indeed, for each $g \in (f)$, say $g = hf$, we have $\mathrm{trop}(V(g)) = \mathrm{trop}(V(h)) \cup \mathrm{trop}(V(f))$, which contains $\mathrm{trop}(V(f))$.
- Chapter 3 concerns the structure of tropical varieties. In particular, it proves that such an object is a polyhedral fan, and that an analogue of Kapranov's theorem holds (the "fundamental theorem of tropical geometry"). But the methods use somewhat technical material from Chapter 2, which we will make a start with next week.
- One inclusion in the fundamental theorem is easy: $\mathrm{trop}(V(I))$ contains the image of $V(I) \subseteq T^n$ under the coordinate-wise valuation map into $\mathbb{R}^n$.
- Remark: in the definition, it is not sufficient to take the intersection over a generating set of I.

- Example: Linear spaces. Let I be generated by linear forms in the variables x_i, and that it does not contain variables, so that $V(I) \subseteq T^n$ is non-empty. The support of a linear form is the set of variables appearing in it. Let S be the collection of supports of linear forms in I. For each inclusion-minimal nonempty support in S there is, up to scaling, a unique linear form in I with that support. Let C be the finite set of such representatives; these are the *circuits* of the linear space.
- Theorem: if I is generated by linear forms, then $\operatorname{trop}(V(I)) = \bigcap_{f \in C} \operatorname{trop}(V(f))$. We'll see a proof later.