TROPICAL GEOMETRY, LECTURE 3

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1. MS §2 Algebraic varieties

- Assume K algebraically closed.
- The *coordinate ring* $K[\mathbb{A}^n]$ of $\mathbb{A}^n$ is $K[x_1, \ldots, x_n]$; the *homogeneous coordinate ring* $K[\mathbb{P}^n]$ of $\mathbb{P}^n$ is $K[x_0, \ldots, x_n]$; and the *coordinate ring* $K[T^n]$ is $K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$.
- Recall that each of these rings is Noetherian (Hilbert's basis theorem): any ideal is generated by finitely many elements.
- $\mathbb{A}^n$ becomes a topological spaces equipped with the *Zariski-topology*, in which a closed subset is of the form $X = V(I) := \{p \in \mathbb{A}^n \mid f(p) = 0 \forall f \in I\}$. Such closed subsets are called *affine varieties*.
- Similarly for $\mathbb{P}^n$ and T^n, where I needs to be spanned by homogeneous polynomials in the case of projective space (Why??). The varieties are called *projective* or *very affine* varieties, respectively. (The term *toric variety*, which you may expect in the last case, is reserved for something else.)
- The topology is Noetherian in each case: any decreasing sequence of closed subsets stabilises. This has as a consequence that each variety can be written uniquely as an irredundant union of irreducible closed subsets.
- Example: the affine variety of pairs (A, B) of 2×2-matrices with $A \cdot B = 0$. This means that the column space of B is contained in the kernel of A. There are three irreducible components: $A = 0$ or $B = 0$ and the rest. The intersection with the torus T^8 has only one irreducible component.
- Can also go back: if X is a subset of affine space, projective space, or the torus, then I_X is the ideal of all polynomials vanishing on X.
- In general, $I_{V(I)} \neq I$; indeed, the left-hand side contains $\sqrt{I}$. Since the field is algebraically closed, equality holds (Hilbert's Nullstellensatz).
- A homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$ is also the ideal of a variety in K^{n+1}; this is called the *cone* over the corresponding projective variety.
- We have dense embeddings $T^n \to \mathbb{A}^n \to \mathbb{P}^n$ (dense, provided the field is infinite). Call the first one i and the second one j. The *affine closure* of a very affine variety $X \subseteq T^n$ is $\overline{i(X)}$. The *projective closure* of an affine variety $X \subseteq \mathbb{A}^n$ is $\overline{j(X)}$. At the level of ideals, we have the following.

Proposition 1.1. *Let $X = V_{T^n}(I)$ be a very affine variety in T^n, where $I \subseteq K[T^n]$. Then $\overline{i(X)} = V_{\mathbb{A}^n}(I \cap K[\mathbb{A}^n])$.*

Proof. For the inclusion $\subseteq$ note that the right-hand side is closed and contains $i(X)$. For $\supseteq$ suppose that p lies in the right-hand side but not in the closure of $i(X)$. Then there is a polynomial $f \in K[\mathbb{A}^n]$ that vanishes identically on $i(X)$ but not on p. By the Nullstellensatz (applied to $K[T^n]$), some power of f lies in I. But this power then also lies in $K[\mathbb{A}^n] \cap I$, hence vanishes on p, a contradiction. □

Similarly: a polynomial $f \in K[x_1, \ldots, x_n]$ has a well-defined *homogenisation* $\tilde{f} \in K[x_0, \ldots, x_n]$. For an ideal I in $K[x_1, \ldots, x_n]$ define I_{proj} as the ideal generated by all $\tilde{f}$ as f runs through I.

Proposition 1.2. *Let $X = V_{\mathbb{A}^n}(I)$ be an affine variety. Then $\overline{j(X)} = V_{\mathbb{P}^n}(I_{\text{proj}})$.*

Proof. The inclusion $\subseteq$ is easy again. For the opposite, let p be in the set on the right and not in the set on the left. There is some homogeneous $f \in K[x_0, \ldots, x_n]$ that vanishes on $j(X)$ but not on p. Then $g := f(1, x_1, \ldots, x_n)$ vanishes on X, hence by the Nullstellensatz some power g^d lies in I. Homogenenising g^d gives a polynomial in I_{proj}, which vanishes on p by assumption. But this polynomial is a nonnegative power of x_0 times f^d, hence f vanishes on p, a contradiction. □

For a lot of tropical geometry, we will be considering only varieties in T^n (this avoids infinite coordinates).

- The ring $K[\mathbb{A}^n]/I_X$, where X is an affine veriety, is called the *coordinate ring* of X. Similarly for very affine varieties (*coordinate ring*) and projective varieties (*homogeneous coordinate ring*).
- A morphism $\phi : X \to Y$ between affine varieties is given by a K-algebra homomorphism $\phi^* : K[Y] \to K[X]$. (This is a contravariant functor between finitely generated K-algebras without nilpotent elements and affine varieties.) Similarly for very affine varieties (for projective varieties the situation is more intricate).
- Important for tropical morphisms are morphisms $\phi : T^n \to T^m$. These are given by homomorphisms $\phi^* : K[T^m] \to K[T^n]$. These must map any invertible element on the left to an invertible element on the right. The invertible elements are precisely the monomials times non-zero scalars. Thus every variable x_j on the left must be mapped to a monomial in the y_i times a non-zero scalar on the right. After dividing by those non-zero scalars (an automorphism of T^n, we end up with a monomial map $\phi^* : x_j \mapsto y_j^{\alpha}$, where $\alpha_j \in \mathbb{Z}^n$. These form the columns of an $n \times m$-matrix over $\mathbb{Z}$.
- Automorphisms $T^n \to T^n$ thus correspond (apart from those scalars) to matrices in $\mathrm{GL}_n(\mathbb{Z})$, i.e., integral square matrices with determinant ± 1.

Lemma 1.3. *The group $\mathrm{GL}_n(\mathbb{Z})$ acts transitively on subgroups $L \subseteq \mathbb{Z}^n$ the quotient by which is torsion-free.*

This follows from the Smith normal form.

Next, we discuss Grassmannians.

- As a set, $G(r, m)$ is the set of r-dimensional subspaces of K^m.
- Represent a U in this set by an $r \times m$-matrix A whose rows are the basis of U.
- Two matrices A and B of full rank r represent the same U iff they differ via left-multiplication with an invertible $r \times r$-matrix.
- Map A to its vector of all $r \times r$-subdeterminants (minors). This is a vector v with $\binom{m}{r}$ entries.
- The map $U \mapsto [v] \in \mathbb{P}^{\binom{m}{r}-1}$ is well-defined and gives an injective map from $G(r, m)$ into this projective space.
- The image is a projective variety cut out by certain quadratic equations called the Plücker relations. This projective variety is called the Grassmannian $G(r, m)$.

- In particular, consider the case $r = 2$ and $m = 4$ in detail.

Next, we prove a lemma relating sets defined by means of the valuation and the Zariski topology.

Lemma 1.4. *Assume K is a non-trivially valued field with valuation group Γ and a section $w \mapsto t^w$ of the valuation. Let $w_1, \ldots, w_n \in \Gamma$ and $\alpha_1, \ldots, \alpha_n \in k^*$. Then the set of all $(y_1, \ldots, y_n) \in (K^*)^n$ with $v(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ is Zariski-dense in $(K^*)^n$.*

Proof. Induction on n. For $n = 0$ the statement is trivially true. Suppose the statement is true for $n - 1$, and let $h \in K[x_1, \ldots, x_n]$ be a non-zero polynomial. We need to show that there is a point y as in the lemma such that $h(y) \neq 0$. Write $h = h_0 + \ldots + h_d x_n^d$ with $h_i \in K[x_1, \ldots, x_{n-1}]$ and $h_d \neq 0$. By induction, there is $y' = (y_1, \ldots, y_{n-1})$ satisfying the lemma such that $h_d(y') \neq 0$. We need to find a y_n satisfying the lemma which is not a root of the non-zero univariate polynomial $h(y', t)$. We prove that there are infinitely many y_n satisfying the conditions of the lemma: pick $z_n \in K$ with $\overline{z_n} = \alpha_n$ and set $y_n := t^{w_n} z_n + r_n$ where $v(r_n)$ is larger than w_n. There are infinitely many such r_n by nontriviality of the valuation. □

2. Some polyhedral geometry

- Definitions: closed convex set in $\mathbb{R}^n$; can be described as the convex hull of its points or, alternatively, as the intersection of the closed halfspaces containing it. (This is a theorem, not a triviality.)
- The *dimension* of a convex set is the dimension of its affine span. Its *relative interior* is its interior when considered as a subset of its affine span.
- Definition: (exposed) face of a convex closed set.
- Special case: closed convex cones; can be described as the positive hull of its intersection with a sphere and or as the the intersection of the closed halfspaces with 0 on their boundary containing it.
- Polyhedron: intersection of *finitely many* closed half-spaces.
- Special case: polyhedral cone, intersection of *finitely many* closed halfspaces with 0 on their boundaries. Equivalently (theorem): the positive hull of finitely many vectors.
- A bounded polyhedron is called a *polytope*. Equivalently (theorem): the convex hull of finitely many points.
- (For these convex sets, all faces are exposed.)
- Definition: polyhedral complexes. Special case: polyhedral fans.
- Definition: common refinement of two polyhedral complexes.
- Definition: Γ-rational polyhedron.
- Definition: normal fan of a polytope P. Cells are labelled by faces F of P, and the cone corresponding to F is

  $$\mathcal{N}_P(F) := \overline{\{w \mid \text{face}_w(P) \supseteq F\}}.$$

  Check that this is a fan!
- Definition: Minkowski sum. Useful facts: normal fan of the Minkowski sum is the common refinement of the normal fans.
- Definition: Newton polytope. Minkowski sum of Newton polytopes is Newton polytope of product.

- The *star* of a polyhedral complex at a cell σ is a polyhedral fan whose cones are labelled by cells τ containing σ. The cone is
$$\overline{\tau} := \{\lambda(x-y) \mid x \in \tau, y \in \sigma\}.$$
Check that this is a polyhedral fan. For instance, $\overline{\tau} \cap \overline{\tau'} = \overline{\tau \cap \tau'}$.
- The regular subdivisions of $v_1, \ldots, v_r \in \mathbb{R}^n$ corresponding to a weight vector $w \in \mathbb{R}^r$: Lift v_i to a point $(v_i, w_i) \in \mathbb{R}^{n+1}$, and take all the faces of the convex hull of the lifted polytope picked out by weight vectors with positive last coordinate. Project these down into P, the convex hull of $v_1, \ldots, v_r$. This gives a polyhedral complex.

3. Homework (hand in by October 9)

Paragraph 2.7: exercises 6, 7, 10, 11.