TROPICAL GEOMETRY, LECTURE 2

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1. Tropical Bézout

Last time we saw properties of tropical plane curves. In fact, they characterise such objects.

Proposition 1.1. *Let Σ be a finite collection of positive-length line segments and half-line rays in $\mathbb{R}^2$, any two of which either do not intersect or else intersect in a common endpoint. Assume that each element of Σ has rational slope and is assigned a positive integral weight such that balancing holds at each endpoint of any element of Σ. Then Σ arises as the set of edges and rays of a tropical curve $V(f)$ for some $f \in \mathbb{R}_\infty[x, y]$.*

In ordinary algebraic geometry, curves of degree d, e intersect in de points, provided that they intersect transversally. A similar result holds in tropical geometry.

Definition 1.2. Let $f, g \in \mathbb{R}_\infty[x, y]$ be non-infinity polynomials, and assume that the line segments and rays in $V(f)$ and those of $V(g)$ intersect transversally. At each intersection point $a \in V(f) \cap V(g)$, let m, n be the weights of the edges of $V(f), V(g)$ through a. The *intersection multiplicity* of $V(f)$ and $V(g)$ at a equals $m \cdot n \cdot |\det(v|w)|$ where $v, w \in \mathbb{Z}^2$ are primitive vectors in the directions of $V(f)$ and $V(g)$ near a. Call this number $m_a(f, g)$.

Proposition 1.3 (Tropical Bézout)**.** *Assume that the Newton polygons of $f, g \in \mathbb{R}_\infty[x, y]$ are the triangles with vertices $(0, 0), (d, 0), (0, d)$ and $(0, 0), (e, 0), (0, e)$, respectively, and that $V(f)$ and $V(g)$ intersect transversally. Then*

$$\sum_{a \in V(f) \cap V(g)} m_a(f, g) = de.$$

Proof. Translate $V(f)$ in a direction $(1, b)$, where we choose $b < 0$ sufficiently generic such that the only combinatorial changes that happen at a time are that a single segment of $V(f)$ passes through a single vertex a of $V(g)$ (and possibly, in the process, coincides with one or two segments of $V(g)$ emanating from that vertex), or vice versa—but in such a way that immediately afterwards, the intersection is again transversal. To analyse what happens in such a change, suppose that an edge of $V(f)$ with primitive vector v passes through a vertex of $V(g)$ with incident segments with primitive vectors $v_1, \ldots, v_k$ and multiplicities $m_1, \ldots, m_k$. Then the segment of $V(f)$ intersects some of these segments before the combinatorial change, say those corresponding to $v_1, \ldots, v_l$, and some of these segments after the combinatorial change, say $v_{l+1}, \ldots, v_p$, and is parallel to the remaining $k - p \in \{0, 1, 2\}$ segments. Now we have, by balancing,

$$\sum_{i=1}^{l} m_i v_i = -\sum_{i=l+1}^{p} m_i v_i - \sum_{i=p+1}^{k} m_i v_i.$$

Here $\det(v|v_i)$ has a constant sign for $i = 1, \dots, l$, the opposite sign for $i = l + 1, \dots, p$, and is zero for $i = p + 1, \dots, k$. This gives that the contribution of the intersection points of $V(f)$ and $V(g)$ near a remains constant in the process.

Now $V(f)$ has d rays in each of the directions north, east, and south-west (counted with weights), and $V(g)$ has e; and these are the only unbounded segments. Here we use the form of the Newton polygon. Hence after translating $V(f)$ as above, we eventually end up with a situation where the only intersection points are those among the d northward rays of $V(f)$ and the e eastward rays of $V(g)$. There are de of these (counted with multiplicities). □

2. MS §2.1 More on valued fields

- Recall the notion of field valuations $v : K \to \mathbb{R}_\infty$.
- $R := \{a \in K \mid v(a) \geq 0\}$ is subring. It has a unique maximal ideal $\mathfrak{m} := \{a \in K \mid v(a) > 0\}$. The field $k := R/\mathfrak{m}$ is called the *residue field* of K.
- For $K = \mathbb{Q}$ with the p-adic valuation, $k = \mathbb{F}_p$; for $K = \mathbb{C}((t))$ with the t-adic valuation, $k = \mathbb{C}$.
- If K is algebraically closed, then so is k.
- K carries a norm determined by $|a| := 2^{-v(a)}$. This induces a metric on K; R is the closed unit sphere around 0 in this norm.
- K with this metric is complete if and only if any series $a_1 + a_2 + \cdots$ in which the $a_i \in K$ tend to zero converges [$\Rightarrow$: the set of partial sums form a Cauchy sequence since $v(a_m + a_{m+1} + \ldots + a_n) \geq \min\{v(a_m), \ldots, v(a_n)\}$; and $\Leftarrow$: if $b_1, b_2, \ldots$ form a Cauchy sequence, then set $a_i := b_{i+1} - b_i$; these form a sequence as above, whose series converges; the limit is the limit of the sequence $(b_i)_i$.]
- $\mathbb{C}((t))$ is complete.
- $\mathbb{C}\{\{t\}\} := \bigcup_{n\in\mathbb{N}} \mathbb{C}((t^{1/n}))$ is the field of *Puiseux series* over $\mathbb{C}$. It is not complete, since for instance $t^{2/1} + t^{5/2} + t^{10/3} + t^{17/4} + \cdots + t^{(n^2+1)/n} + \cdots$ does not converge.

Proposition 2.1. *$K := \mathbb{C}\{\{t\}\}$ is algebraically closed.*

- When a field k has characteristic p, the field $k\{\{t\}\}$ is not algebraically closed; see below.
- This motivates the following definition. Let $G \subseteq \mathbb{R}$ be any *divisible subgroup*, and let $k((G))$ be the set of series $\sum_{i\in A} c_i t^i$ where A is a *well-ordered subset* of G. These form a field, and if k is algebraically closed, it is an algebraically closed, valued extension of $k((t))$ (the fact that it is algebraically closed is nontrivial).
- Example: take $G = \mathbb{Q}$ and k and field of characteristic p. The polynomial $x^p - x - t^{-1} \in k((t))$ has roots $(t^{-1/p} + t^{-1/(p^2)} + \cdots) + c$ for each $c \in \mathbb{F}_p$. These lie in $k((G))$ but not in $k\{\{t\}\}$ as the denominators of the exponents are unbounded. Hence the latter field is not algebraically closed.

It will be useful to have, in general, a section to a valuation.

Lemma 2.2. *Suppose that (K, v) is a valued field such that for each $a \in K$ with $v(a) = 0$ and for each positive integer n, there there exists exists an element $b \in K$ with $b^n = a$. Then there is a map $\psi : \Gamma \to K$ such that $\psi(a + b) = \psi(a)\psi(b)$ and $v(\psi(a)) = a$.*

The proof of this lemma uses the following general fact.

Lemma 2.3 (Divisible groups are direct summands.). *Let $(A, +, 0)$ be an Abelian group and $U \subseteq A$ a* divisible subgroup*, i.e., a subgroup such that for all $nU = U$ for all positive integers n. Then A has a subgroup W such that $A = U \oplus W$.*

Proof. Consider all subgroups $W \subseteq A$ that intersect U only in 0. By Zorn's lemma, there is a maximal subgroup W with this property. We claim that $U + W = A$. Indeed, suppose that $a \in A \setminus (U + W)$. By maximality of W, we have $na + w = u \in U \setminus \{0\}$ for some $n > 0$. Pick n minimal with this property; this is the order of the image of a in $A/(U + W)$. Since $a \notin U + W$, we have $n > 1$. Pick $u' \in U$ such that $nu' = u$.

Now $W + \langle u' - a\rangle$ is strictly larger than W, and we claim that it does not intersect U in any element u'' other than 0. Indeed, such an element would be of the form $w'' + m(u' - a)$ with m a multiple of n, so $w'' + (m/n)w \in W$. □

Proof of Lemma 2.2. The map $v : (K^*, \cdot) \to (\mathbb{R}, +)$ is a group homomorphism whose kernel is divisible by assumption. So by the preceding lemma, we can write $K^* = \ker(v) \cdot W$, where W is a subgroup intersecting $\ker(v)$ trivially. Now v restricts to an isomorphism $W \to \mathrm{im}(v)$, so we can take ψ equal to its inverse. □

Remark 2.4. Many of our examples will be over the field over Puiseux series with $\mathbb{C}$ coefficients. However, a general Puiseux series cannot be represented on a computer, so in practice the field elements we work with will be rational functions in $\mathbb{Q}(t)$.

There is more material on valued extensions, with references to the literature, but this is all we will need.

3. MS §2 Algebraic varieties

- Fix a field K.
- Our algebraic varieties will (almost) always be embedded in one of the following three ambient spaces:
  1. $\mathbb{A}^n = \mathbb{A}^n_K = K^n$, the *$n$-dimensional affine space* with coordinates $x_1, \ldots, x_n$;
  2. $\mathbb{P}^n = \mathbb{P}^n_K = (K^{n+1} \setminus \{0\})/\sim$, the n-dimensional *projective space* where two non-zero vectors $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ iff one is a scalar multiple of the other. We write $[x_0 : \ldots : x_n]$ for the equivalence class of $(x_0, \ldots, x_n)$.
  3. $T^n = T^n_K = (K^*)^n$, the n-dimensional *torus*.
- The *coordinate ring* $K[\mathbb{A}^n]$ of $\mathbb{A}^n$ is $K[x_1, \ldots, x_n]$; the *homogeneous coordinate ring* $K[\mathbb{P}^n]$ of $\mathbb{P}^n$ is $K[x_0, \ldots, x_n]$; and the *coordinate ring* $K[T^n]$ is $K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$.
- Recall that each of these rings is Noetherian (Hilbert's basis theorem): any ideal is generated by finitely many elements.
- $\mathbb{A}^n$ becomes a topological spaces equipped with the *Zariski-topology*, in which a closed subset is of the form $X = V(I) := \{p \in \mathbb{A}^n \mid f(p) = 0 \forall f \in I\}$. Such closed subsets are called *affine varieties*.
- Similarly for $\mathbb{P}^n$ and T^n, where I needs to be spanned by homogeneous polynomials in the case of projective space (Why??). The varieties are called

projective or *very affine* varieties, respectively. (The term *toric variety*, which you may expect in the last case, is reserved for something else.)

- The topology is Noetherian in each case: any decreasing sequence of closed subsets stabilises. This has as a consequence that each variety can be written uniquely as an irredundant union of irreducible closed subsets.
- Example: commuting variety of pairs (A, B) of 2×2-matrices (over $\mathbb{C}$). Contains as an open set the set where A has two distinct eigenvalues, which is contained in $\{(gDg^{-1}, gEg^{-1}) \mid g \in \mathrm{GL}_2 \text{ and } D, E \text{ diagonal}\}$. This set, and hence its closure, are irreducible. Why is this closure the entire commuting variety? It certainly contains the pairs where A is a scalar multiple of the identity, by taking D equal to A and using that the diagonalisable matrices gEg^{-1} are dense the space of 2×2-matrices B. So only the case left is where A is of the form
$$A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$
In this case, $AB = BA$ forces that $B = sA + tI$, so if we take a sequence of g_i, D_i such that $g_i D_i g_i^{-1}$ converges to A, then with $E_i := sD_i + tI$ the matrix $g_i E_i g_i^{-1}$ converges to B. Thus the commuting variety is irreducible. (It also is for larger matrices.)
- Example: variety of pairs (A, B) with $A \cdot B = 0$. This means that the column space of B is contained in the kernel of A. There are three irreducible components: $A = 0$ or $B = 0$ and the rest.
- Can also go back: if X is a subset of affine space, projective space, or the torus, then I_X is the ideal of all polynomials vanishing on X.
- In general, $I_{V(I)} \neq I$; indeed, the left-hand side contains $\sqrt{I}$. When the field is algebraically closed, equality holds (Hilbert's Nullstellensatz).
- A homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$ is also the coordinate ring of a variety in K^{n+1}; this is called the *cone* over the corresponding projective variety.
- We have dense embeddings $T^n \to \mathbb{A}^n \to \mathbb{P}^n$.
- For a lot of tropical geometry, we will be considering only varieties in T^n (this avoids infinite coordinates).