TROPICAL GEOMETRY, LECTURE 13

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1. Reminder: divisor class groups of curves

For a smooth, projective curve X over an algebraically closed field K:

- $\mathrm{Div}(X) := \mathbb{Z}X$ is the group of divisors

- Notation: $D(x)$ for the coefficient of $x \in X$ in D.
- Degree map $\deg : \mathrm{Div}(X) \to \mathbb{Z},\ D \mapsto \sum_{x\in X} D(x)$ and $\mathrm{Div}_d(X) := \{D \in \mathrm{Div}(X) \mid \deg(D) = 0\}$.
- For a rational function f on X, $\div f \in \mathrm{Div}(X)$ is defined as $(\div f)(x) =$the order of vanishing of f at x. (Positive for roots, negative for poles.)
- Example: $X = \mathbb{P}^1$, $f = \frac{x}{x^2-2x+1} = \frac{xy}{x^2-2xy+y^2}$ has $\div f = (0 : 1) - 2(1 : 1) + (1 : 0)$. Has degree 0; this is a general fact for rational functions.
- The divisors of the form $\div f$ are called *principal*; they form a subgroup $\mathrm{Prin}(X)$ of $\mathrm{Div}_0(X)$.
- The quotient $\mathrm{Prin}(X) := \mathrm{Div}(X)/\mathrm{Prin}(X)$, the *divisor class group*, still has the degree homomorphism to $\mathbb{Z}$. The kernel $\mathrm{Prin}_0(X)$ has the structure of a compact, abelian, algebraic group of dimension the *genus* of X.
- We write $D \sim E$ if $D - E \in \mathrm{Prin}(X)$ and say that they are *linearly equivalent*; write $[D]$ for the class of D modulo $\mathrm{Prin}(X)$.
- Example: for $\mathbb{P}^1$ the group $\mathrm{Div}_0(X)/\mathrm{Prin}(X)$ is trivial; for an elliptic curve X with a distinguished point $p_0 \in X$ (so that X is a group with neutral element p_0), it is isomorphic to X via the map that sends $p \mapsto [p - p_0]$.

  [Part of the verification is this: embed X in $\mathbb{P}^2$ with homogeneous coordinates $(x : y : z)$ as a cubic curve such that p_0 is the only point of X at infinity. Then, in the group structure of the elliptic curve X, $p + q + r$ equals the neutral element p_0 if and only if they are the intersection points (with multiplicities) of a line in $\mathbb{P}^2$ with X. In that case, let l be the defining linear form of that line, and set $f := l/z$. We find that $\div(f) = -3p_0 + p + q + r$, and hence that the sum of the divisors $[p - p_0] + [q - p_0] + [r - p_0] = [p + q + r - 3p_0] = [\div(f)] = 0$.
- $E \in \mathrm{Div}(X)$ is called *effective* if $E(x) \geq 0$ for all x. We then write $E \geq 0$, and for general divisors we write $F \geq D$ iff $F - D \geq 0$.
- Given $D \in \mathrm{Div}(X)$, the set $|D| := \{E \in \mathrm{Div}(X) \mid E \sim D \text{ and } E \text{ effective}\}$ is called the *complete linear system* of D. It has a bijection with the projective space of the vector space $\{f \text{ rational } \mid \div(f) \geq -D\}$. Note that this is, indeed, a vector space, since $(\div(f + g))(x)$ is at least the minimum of $(\div f)(x)$ and $(\div g)(x)$. (That this is a bijection uses that a rational function is determined up to a scalar by its poles, which is one of the reasons why we need to assume that X is projective). This vector space is has finite dimension $r(D) + 1$, where $r(D)$ is called the *rank* of D.

- Tropicalised curves (say, in T^n) are one-dimensional polyhedral complexes, with natural edge lengths (lattice length).
- Rational functions on T^n correspond to differences of tropical polynomials.

2. Divisor class groups of graphs

- Let X be a connected metric graph, obtained by gluing, in some manner, finite-length closed intervals along their endpoints.
- Elements of $\mathrm{Div}(X) = \mathbb{Z}X$ are called *divisors* on X, notation $D(x)$ as before.
- $\deg : \mathrm{Div}(X) \to \mathbb{Z}X$ is the degree map, $\mathrm{Div}_d(X)$ the set of degree-d divisors.
- $f : X \to \mathbb{R}$ is called *(tropical) rational* if it is continuous and linear with integral slopes outside a finite number of points.
- Rational functions on X form a group $M(X)$ with respect to addition (tropical multiplication).
- $\div f \in \mathrm{Div}(X)$ is defined by $(\div f)(x) =$ the sum of the incoming slopes of f at x. If x has valency 2, then rational functions locally concave at x have a root there and rational functions locally convex at x have a pole. $\div f$ is the *principal divisor associated to* f. Write $\mathrm{Prin}(X) := \{\div(f) \mid f \in M(X)\}$.
- Note that $\div f$ has degree 0: for every segment where f is linear, its incoming slope at one endpoint is minus its incoming slope at the other endpoint.
- $\div$ is a group homomorphism from $(M(x), \odot)$ to $\mathrm{Div}_0(X)$. So its image $\mathrm{Prin}(X)$ is a subgroup of $\mathrm{Div}_0(X)$.
- Note also that $\div f$ determines f up to a tropically multiplicative (i.e., additive) scalar.
- The group $\mathrm{Prin}(X) := \mathrm{Div}(X)/\mathrm{Prin}(X)$ is called the *divisor class group* of X. The class of D is denoted $[D]$. $\mathrm{Prin}(X)$ has the degree homomorphism into $\mathbb{Z}$, whose kernel is $\mathrm{Prin}_0(X) = \mathrm{Prin}(X) \cap \mathrm{Div}_0(X)$.
- Let g be the first Betti number of X (i.e., the number of cuts you need to make to make X into a metric tree). Theorem: the group $\mathrm{Prin}_0(X)$ is isomorphic to $(S^1)^g$ as a topological group. We will also call g the *genus* of X.

3. Dhar's burning algorithm

- Fix a point $q \in X$. This algorithm chooses a unique D_q of a divisor $D \in \mathrm{Div}(X)$, with support "as close as possible" to q. It is called the *q-reduced representative* of $[D]$ and has the following properties:
  1. $D_q \sim D$;
  2. D_q is effective outside q;
  3. Any nonempty closed subset $Y \subseteq X$ not containing q has at least one boundary point y where $D_q(y)$ is strictly less than the number of edges emanating from y into $X \setminus Y$.
- I'll describe it for a D that is already effective outside q. Think of D as putting $D(x) \geq 0$ chips on finitely many points x outside q, and a possibly negative number of chips at q.
  1. Initialise $F := D$.
  2. Start burning a small open neighbourhood of q in X containing no chips except possibly at q; chips (or negative chips) of F at q will

always remain there. We agree that the burned set will always be open, connected, and contain q.
(3) If the fire arives at a point $p \in X$ from more directions than F has chips at p, then the fire passes through p.
(4) If all of X burns, then output $D_q := F$ and stop. Otherwise, let $Y \subseteq X$ be the closed set that was not burned, and U the open set that was burned.
(5) Let $d > 0$ be the smallest distance from a valency->2 vertex in U to Y. Observe that there are no chips on U at distance smaller than d to Y.
(6) Let $Z \subseteq X$ be the closed set of points at distance $\geq d$ to Y.
(7) Now let f be the rational function that is d on Z, 0 on Y, and linear with slope 1 on all line segments connecting Z and Y.
(8) So $\div(f)$ is supported on the boundary of Z (where it has positive coefficients) and the boundary of Y (where it has negative coefficients). The coefficient of $\div(f)$ at a boundary point y of Y is the number of edges emanating from y into U, hence at most the number of chips of F at y since burning stopped at y.
(9) Hence $D + \div(f)$ is effective and has its chips closer to q. Now go back to step 2.

- This algorithm terminates, and the output has the properties above. (For the last property: if Y did not have such a boundary point, it would not burn!)
- Lemma: there is only one divisor linearly equivalent to a given D with the properties above.

  [Given two of them, say E and F, let $f \in M(X)$ be such that $E + \div(f) = F$. Let Y be the set where f attains its minimal value; this is a closed set. If $q \notin Y$, then, by the assumption on E, Y has a boundary point y where $E(y)$ is less than the number of edges emanating from y into $X \setminus Y$. Along each of these edges the incoming slope of f is negative, so $F(x) = (\div f)(x) + E(x) < 0$, a contradiction. Hence f is minimal at q. By applying the same reasoning to $-f$, with the roles of F and E reversed, f is also maximal at q. Hence f is constant.]
- Example: $\mathrm{Prin}_0(X)$ of a circle X with base point q, considered as neutral element of its group of rotations, is isomorphic to X via the map that sends p to $[p - q]$.

4. Riemann's inequality

- In the algebro-geometric setting, we have $r(D) \geq \deg(D) - g$ (the difference is $r(K - D) + 1 \geq 0$, where K is the *canonical divisor*).
- (Of course, this is only an interesting inequality if $\deg(D) \geq g$.)
- In particular, this means that given any effective divisor E of degree $\deg(D) - g$, then there exists a rational function f such that not only $\div(f) \geq -D$ but indeed $\div(f) \geq -D + E$. (Vanishing to the order e at a point $p \in X$ imposes at most e linearly independent conditions on $|D|$. If the number of conditions does not exceed the dimension $r(D)$ of this projective space, then there exists a point with those conditions.)
- We will now prove the analogue for graphs.

- Theorem: for a metric graph X of genus g, if D is any divisor with $\deg(D) \geq g$, and if E is any effective divisor of degree $\deg(D)-g$, then $D-E$ is linearly equivalent to an effective divisor.
- In fact, we will prove the full Riemann-Roch theorem, following Matt Baker's blog. It says the following.
- Definition: let K be the divisor with $K(x) =$ valency of x minus 2. Check that $\deg(K) = 2g - 2$.
- Definition: for D any divisor on X, let $r(D)$ denote the maximal $r \in \mathbb{Z}_{\geq -1}$ such that for each effective divisor E of degree r the divisor $D - E$ is equivalent to an effective divisor.
- Theorem (Riemann-Roch): $r(D) - r(K - D) = \deg(D) - g + 1$.
- Proof from Baker's blog. Every orientation O of X yields a divisor D_O with $D_O(x)$ =number of incoming edges at x-1. This has degree $g - 1$. If O' is the opposite orientation, then $D_O + D_{O'} = K$.
- For a continuation see `https://mattbakerblog.wordpress.com/2014/01/12/reduced-divisors-and-` and start reading at Lemma 2. (Note that there, things are written down for ordinary rather than metric graphs, but the proofs are the same.)