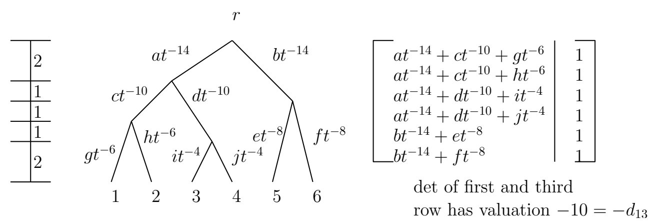
TROPICAL GEOMETRY, LECTURE 12

JAN DRAISMA

Grassmannians of two-spaces

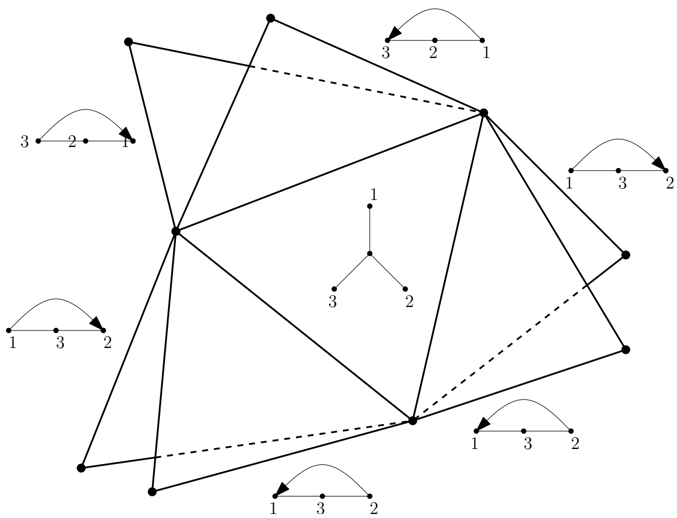
- In this section, based on work by Speyer and Sturmfels, we tropicalise the variety $X := \widehat{G(2,n)} \cap T^{\binom{n}{2}}$ with coordinates $(p_{ij})_{i<j}$ defined by the vanishing of the three-term Plücker polynomials
$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk}$$
for $i < j < k < l$.
- Clearly, for $w \in \mathrm{Trop}(X)$ it is necessary that each minimum $\min\{w_{ij} + w_{kl}, w_{ik} + w_{jl}, w_{il} + w_{jk}\}$ is attained at least twice. We will prove that this also suffices, so that the Plücker polynomials form a tropical basis.
- A good source of tuples w satisfying the above is obtained by taking $w = -d$ where d is a tree metric. Up to the lineality space, these are all. Indeed: consider $d := -w + N(1, \ldots, 1)$ where N is so large that d satisfies all triangle inequalities. Then d satisfies the four-point condition, hence is a tree metric. And the vector $(1, \ldots, 1)$ lies in the lineality space of X.
- So it suffices to prove that if $w = -d$ where d is a tree metric corresponding to the metric tree T and a labelling $[n] \to T$, then $w = v(p)$ for some point $p \in X$.
- Choose a root r on the interior of one of the edges e_0, and add to the edge lengths leading to leaves positive real numbers such that the path from r to any of the leaves $1, \ldots, n$ is equal to a fixed number $W > 0$. This is the "balanced" case.
- For each edge e, let u_e be the length of the path starting with e to any of the leaves below it. (this doesn't depend on the leaf). Thus the two edges e leading to r have u_e equal to W.
- Choose generic elements $c_e \in K$ of valuation 0. For each leaf i let q_i be the sum of the terms $c_e t^{-2u_e} \in K$ where e runs over the edges on the path from i to r (inclusive).
- Now set $p_{ij} := q_i - q_j \in K$. The paths from i and j to r meet for the first time at some vertex t. The contribution of the edges between r and t cancel in the above expression. Consequently, by genericity of the coefficients, the valuation of p_{ij} equals $v(p_{ij}) = -2u_e$, where e is any of the two edges leading down from t. And this equals $w_{ij} = -d_{ij}$.



- The general, unbalanced case follows by acting with a suitable from $T^n \subseteq T^{\binom{n}{2}}$, $(x_i)_i \mapsto (x_i x_j)_{i<j}$.

Tropical products of distance matrices

- Denote by $\odot$ tropical matrix multiplication.
- Theorem: the set $\mathcal{D} := \{D_1 \odot \cdots \odot D_k \mid k \in \mathbb{Z}_{\geq 1}, D_i\ n \times n\text{-metric}\}$ is the support of some finite polyhedral fan of dimension $\binom{n}{2}$.
- Interpretation of entry in the product at (i,j): first do a step in metric 1, then in metric 2, etc., to get as efficiently as possible from i to j.
- Example for $n = 3$:



- Finiteness: can bound k as follows. To go from i to j you will need never use more than $(n-1)$ of the metrics. Hence you can delete all but $n^2(n-1)$ of the metrics, while the result stays the same. Hence the set in the theorem does not change if we restrict k to be at most (or exactly) $n^2(n-1)$. Hence $\mathcal{D}$ is the (union of) images of finitely many polyhedra under piecewise linear maps, hence the support of some finite polyhedral complex.
- Lower bound on the dimension: $\binom{n}{2}$ is the dimension of the cone of metric matrices.
- So the upper bound remains to be proved.

  First, take a $D = (d_{ij})_{ij}$ where all triangle inequalities hold strictly and all $d_{ij} > 0$ for $i \neq j$. Construct the skew-symmetric matrix X over $K := k((\mathbb{R}))$ (formal power series in t over k with arbitrary well-ordered supports in $\mathbb{R}$).

  Then $I - X$ is invertible with inverse $I + X + X^2 + \dots$ (for this series to make sense we use $d_{ij} > 0$) for $i \neq j$. Set $g := (I+X)(I-X)^{-1}$ (Cayley transform). This is orthogonal:

$$g^T g = (I - X)^{-T}(I + X)^T(I + X)(I - X)^{-1}$$
$$= (I + X)^{-1}(I - X)(I + X)(I - X)^{-1}.$$

Moreover, by the strict triangle inequalities, $v(g) = D$.

Hence the cone of distance matrices is contained in $\text{trop}(O_n \cap T^{n\times n})$ (the latter set is closed, so don't need strict ineqs).

Similarly, for k metric matrices $D_1, \ldots, D_k$ with strict ineqs, construct $X_1, \ldots, X_k$ and $g_1, \ldots, g_k$ as above, except multiply the powers of t_{ij} by suitable constants to achieve that $v(g_1 \cdots g_k) = D_1 \odot \cdots \odot D_k$ (no cancellation). Since O_n is closed under the ordinary matrix product, we find $\mathcal{D} \subseteq \text{trop}(O_n \cap T^{n\times n})$.

Now O_n is defined by $\binom{n+1}{2}$ equations and hence has dimension at least $\binom{n}{2}$ by the principal ideal theorem. The tangent space at I has dimension at most equal to $\binom{n}{2}$, hence O_n is smooth at I and the irreducible component of O_n containing I has exactly that dimension. But then so do all irreducible components of O_n: if $h \in O_n$, then left multiplication by h yields an automorphism of O_n that maps the component containing I to the component containing h (this is a general fact about algebraic groups over algebraically closed fields: they are smooth and equidimensional).

It follows that $\dim \text{trop}(O_n \cap T^{n\times n}) = \dim O_n \cap T^{n\times n} = \binom{n}{2}$, where the first inequality follows from the structure theorem. This proves that $\mathcal{D}$ has at most that dimension.

- We do not know whether $\mathcal{D}$ is pure, or whether $\text{trop}(O_n \cap T^{n\times n})$ is closed under $\odot$ (we used this only in a special case).
- Incidentally, O_n also has an interesting, (nontropical) application to the extremal combinatorics of gossiping.

1. Tropicalising a linear space, bis

- Assume $v(K) = \mathbb{R}$ and k infinite. Let $X \subseteq K^n$ be a linear space. We will compute $v(X) \subseteq \mathbb{R}^n_\infty$.
- $v(X)$ is closed under tropical addition: if $a, b \in v(X)$ then we can pick a $\lambda \in K$ of valuation zero such that $v(X) \ni v(a + \lambda b) = v(a) \oplus v(b)$. (For the i-th entry, this is automatic if $v(a_i) \neq v(b_i)$. If they're equal, we should pick λ such that $1 + \overline{\lambda b_i/a_i} \neq 0$ in k. This is where we use that k is infinite.)
- $v(X)$ also contains $(\infty, \ldots, \infty)$ and is closed under tropical scalar multiplication. Together, these three statements mean that they form a $\mathbb{R}_\infty$-subsemimodule of $\mathbb{R}^n_\infty$.
- Let $x_1, \ldots, x_N$ be the vectors in X of distinct, minimal nonempty supports (sets of positions where they are nonzero). These are called the co-circuits of the matroid defined by X.
- Theorem (Yu-Yuster): $v(X)$ equals the $\mathbb{R}_\infty$-subsemimodule M generated by $v(x_1), \ldots, v(x_N)$.

  [$\supseteq$ follows from the above.

  $\subseteq$: Prove by induction on the support of $x \in X$ that $v(x) \in M$. Suppose that this holds for all vectors with support strictly contained in that of x. Choose j such that the support S of x_j is contained in that of x. We want to choose $\lambda \in K$ such that $y := x - \lambda x_j \in X$ has strictly smaller support than x and moreover $v(x) = v(y) \oplus (v(\lambda) \odot v(x_j))$. For this, choose $i \in S$ such that $v(x_i) - v((x_j)_i)$ is minimal and set $\lambda := x_i/(x_j)_i$. Then the support of y does not contain i, both $v(y)$ and $v(\lambda x_j)$ are componentwise at least $v(x)$, and where the first one is strictly larger than $v(x)$, the second one equals $v(x)$. Hence $v(x) = v(y) \oplus (v(\lambda) \odot v(x_j))$, as claimed.]