TROPICAL GEOMETRY, LECTURE 10

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The structure theorem

- Last time: the fundamental theorem.
- Notation: for a variety X defined over K and L a field extension, let $X(L)$ be its set of point with coordinates over L.
- Consequence: the set $\overline{v(X(L))}$ is the same, and equal to $\operatorname{trop}(X)$, for any algebraically closed, non-trivially valued field extension L of an arbitrary valued field K.
- Stronger version: let K be a non-trivially valued, algebraically closed field, and $X \subseteq T^n$ an irreducible subvariety defined over K. Then for each $w \in \operatorname{Trop}(X) \cap (v(K^*)^n)$ the set of $p \in X$ with $v(p) = w$ is Zariski-dense in X. We saw this for the hypersurface case (Kapranov's theorem), and the induction step that proved the fundamental theorem is consistent with this.
- Consequence: let $\phi : T^n \to T^m$ be a torus homomorphism, let $X \subseteq T^n$ be a subvariety defined over K, and set $Y := \overline{\phi(X)}$. Then the linear map $\operatorname{trop}(\phi) : \mathbb{R}^n \to \mathbb{R}^m$ maps $\operatorname{trop}(X)$ onto $\operatorname{trop}(Y)$.

  [We have already seen (and used!) "into". But now it also follows like this: pick an algebraically closed, valued field extension L of K with surjective valuation, so that $\operatorname{trop}(X) = v(X(L))$. Then $\operatorname{trop}(\phi)\operatorname{trop}(X) \subseteq v(\phi(X(L))) \subseteq v(Y(L)) = \operatorname{trop}(Y)$.

  For the converse, first reduce to the case where X is irreducible. Then so is Y. For a point $w \in \operatorname{Trop}(Y)$ the set of $q \in Y(L)$ with $v(q) = w$ is dense. Hence it intersects $\phi(X)$. (Indeed, the image of a variety under a morphism is a "constructible set": a finite union of locally closed subsets. In particular, $\phi(X)$ contains an open, dense subset of Y. Such a subset intersects any dense set.)]
- Example (Grassmannians of 2-spaces): Let V be the subvariety of $T^{\binom{m}{2}}$ with coordinates $x_{ij} = -x_{ji}, i \neq j$ given by the linear equations $x_{ij} + x_{jk} + x_{ki} = 0$. (This is the image of the rational map from K^m that maps $(t_1, \ldots, t_m)$ to $(x_{ij} = t_i - t_j)_{i<j}$.)

  Set $X := T^m \times V \subseteq T^{m+\binom{m}{2}}$ and consider the torus homomorphism $\phi : T^{m+\binom{m}{2}} \to T^{\binom{m}{2}}$ that maps $((s_1, \ldots, s_m), (x_{ij})_{ij})$ to $(s_i s_j x_{ij})_{ij}$. The image of ϕ is dense in the affine cone $\widehat{G(2,m)}$ over the Grassmannian $G(2,n)$, and hence $\operatorname{trop}(\widehat{G(2,m)} \cap T^{\binom{m}{2}}) = \operatorname{trop}(\phi)\operatorname{trop}(X)$. The right-hand side is the image of $\mathbb{R}^m \times \operatorname{trop}(V)$ under the map $(s,x) \mapsto (s_i + s_j + x_{ij})_{ij}$.
- More general example: if $Y \subseteq T^n$ obtained from $X \subseteq T^n$ as the image $\overline{\phi(T^m) \cdot X}$, where $\phi : T^m \to T^n$ is a torus homomorphism and where

the action is component-wise multiplication, then $\operatorname{trop}(Y) = \operatorname{trop}(X) + \operatorname{trop}(\phi)\mathbb{R}^m$.

- Theorem: (the *structure theorem*): The tropicalisation $\operatorname{trop}(X)$ of an irreducible algebraic variety $X \subseteq T_K^n$ of dimension d is the support of a pure, d-dimensional, $v(K^*)$-rational polyhedral complex, which is connected in codimension one, and in which moreover all d-dimensional cells are given a nonnegative integer as multiplicity in such a way that "balancing" holds.
- *Pure* means that all maximal cells have the same dimension d.
- *Balancing* means the following: suppose that τ is a cell of dimension $d-1$, that $\sigma_1, \ldots, \sigma_k$ are the d-dimensional cells containing τ, and that $m_1, \ldots, m_k$ are their multiplicities. Translate such that τ has 0 in its relative interior, and let $L \subseteq \mathbb{R}^n$ be the $(d-1)$-dimensional subspace of $\mathbb{R}^n$ spanned by τ. Since τ is $v(K^*)$-rational, L is the span of the (saturated) lattice $M := L \cap \mathbb{Z}^n$. Similarly, we obtain rank-d lattices $M_1, \ldots, M_k$ containing M for the σ_i. Each quotient $M_i/M \cong \mathbb{Z}$ is a one-dimensional lattice, and exactly one of its two generators has a representative $v_i \in M_i$ such that $\epsilon v_i \in \sigma_i$ for small $\epsilon > 0$. Then balancing says that
$$m_1 v_1 + \ldots + m_k v_k \in M.$$
- *Connected in codimension one* means that for any two d-dimensional cells σ, σ' there is a sequence $\sigma = \sigma_0, \ldots, \sigma_k = \sigma'$ of maximal cells such that any two consecutive cells have a $(d-1)$-dimensional facet in common.
- The construction of the multiplicities is somewhat technical, and so is the proof of balancing. Let's only discuss the simple hypersurface case, where $X \subseteq T^n$ is the zero set of a single polynomial f.

  [In this case, $\operatorname{trop}(X)$ is dual to the regular subdivision of the Newton polytope of f with lifting function $\alpha \mapsto v(c_\alpha)$. Thus, the $(n-1)$-dimensional cells of $\operatorname{trop}(X)$ correspond bijectively to edges in this decomposition. Define their multiplicities as the lattice lengths of the corresponding edges (number of lattice points minus one).

  Now fix an $(n-2)$-dimensional cell τ, and translate such that τ has 0 in its relative interior. This τ corresponds to a polygon P in the regular subdivision. By a lattice automorphism we may transform the lattice M corresponding to τ into the lattice $\mathbb{Z}^{n-2}$ where the first two coordinates are zero. This transforms P, which lives in the dual space, into a lattice polygon with vertices in $\mathbb{Z}^2$, the annihilator of the $\mathbb{Z}^{n-2}$ corresponding to τ. Let $a_1, \ldots, a_k \in \mathbb{Z}^2$ be the vertices of this polygon read off in counterclock-wise order, so that $f_i := a_{i+1} - a_i$ are the vectors corresponding to edges of P. The span of the corresponding $(n-1)$-dimensional cell σ_i is the annihilator $f_i^\perp$ in the dual $\mathbb{Z}^n$. Write $f_i = m_i f_i'$, where $f_i' = (a_i, b_i, 0, \ldots, 0) \in \mathbb{Z}^2$ has coprime coordinates and where m_i is the lattice length of f_i, hence the multiplicity of σ_i. Then, near 0, σ_i agrees with the cone spanned by τ and $(-b_i, a_i, 0, \ldots, 0)$. Hence we may take the representative $v_i := (-b_i, a_i, 0, \ldots, 0)$. The fact that $f_1 + \ldots + f_k = 0$ now translates into the balancing condition that $m_1 v_1 + \ldots + m_k v_k \in \mathbb{Z}^{n-2}$.]
- The connectedness is highly nontrivial, even if there is a relatively simple reduction to the case of a curve in three-space. We omit the proof.
- So we are left with proving that $\operatorname{trop}(X)$ is pure of dimension d if X is irreducible (or just equidimensional) of dimension d. Last week we saw that

all cells in the Gröbner complex of I_{proj} have dimension at most d, when regarded in $\mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$. Now we need to prove that the maximal cells have exactly that dimension. We first prove a weaker statement: namely, that if $\dim X > 0$, then $\text{trop}(X)$ has a positive dimensional cell.

- Lemma: suppose that $Y \subseteq T^n$ is a closed subvariety such that $\text{trop}(Y)$ is a finite subset of $\mathbb{R}^n$. Then Y has dimension 0, i.e., Y is a finite set of points.

  [Induction on n. For $n = 1$ the statement is clear since if Y does not have dimension zero, then $Y = T^1$ and $\text{trop}(Y) = \mathbb{R}$. If the statement holds for $n-1 \geq 1$, then consider a torus homomorphism $\psi : T^n \to T^{n-1}$ such that $Z := \psi(Y)$ is closed and of the same dimension as Y. Then $\text{trop}(Z) = \text{trop}(\psi(Y)) = \text{trop}(\psi)\text{trop}(Y)$ is finite, and hence $\dim Y = \dim Z = 0$ by the induction hypothesis.]
- Lemma: let $X \subseteq T^n$ be an irreducible subvariety with ideal $I = I_X$ and $w \in \text{trop}(X)$. Then $\dim V(\text{in}_w I) = \dim X =: d$.
- Last week we used only the inequality $\leq$, which does not need the irreducibility of X. Let's see in an example why we need irreducibility. Consider the ideal $I = \langle x+y+1\rangle \cap \langle x-t^2, y-3t\rangle = \langle (x+y+1)(x-t^2), (x+y+1)(y-3t)\rangle$ and set $X := V_{T^n}(I)$. Then $\text{trop}(X)$ is the union of the single point $(2,1)$ and the tropicalisation of a line. Now $\text{in}_{(2,1)} I \supseteq \langle x-1, y-3\rangle$, and since this is a maximal ideal and $(2,1) \in \text{trop}(X)$, equality holds. So $V(\text{in}_{(2,1)} I)$ is just the single point $(1,3)$.

  [Proof of the lemma:

  The ideal $\text{in}_w I_{\text{aff}}$ is the image of $\text{in}_{(0,w)} I_{\text{proj}}$ under the map $x_0 \mapsto 1$. The Krull dimension of $K[P^n]/I_{\text{proj}}$ is $d+1$, hence so is (by the work of Chapter 2) the Krull dimension of $k[P^n]/\text{in}_{(0,w)} I_{\text{proj}}$.

  But now we will need the more precise statement, namely, that *all irreducible components* of $V(\text{in}_{(0,w)} I_{\text{proj}}) \subseteq \mathbb{A}_k^{n+1}$ have dimension $d+1$; here we use irreducibility of X. (In Chapter 2, we were too lazy to go through the proof of this statement.) By intersecting with the hyperplane $x_0 = 1$ we loose the components contained in the hyperplane $x_0 = 0$, and the remaining components have dimension one lower than $d+1$, i.e., d (use the Principle Ideal Theorem). Hence, $V_{\mathbb{A}^n}(\text{in}_w I_{\text{aff}})$ is equidimensional of dimension d. But then so is $V_{T^n}(\text{in}_w I)$, (which, by assumption, is not empty).]
- Proof of pureness 1:

  Let $\sigma \subseteq \mathbb{R}^n \cong \mathbb{R}^{n+1}/\mathbb{R}(1,\ldots,1)$ be a maximal cell in the Gröbner complex of I_{proj}. Set $k := \dim \sigma$ and let w be in the relative interior of σ. After a torus automorphism, we may assume that the affine span of σ equals $w+L$ with $L = \langle e_1, \ldots, e_k\rangle$. For all $u \in L$ we have $\text{in}_u \text{in}_w I = \text{in}_{w+\epsilon u} I = \text{in}_w$ if $\epsilon > 0$ is sufficiently small, so (as in one of last week's proofs) $\text{in}_w I$ is $\mathbb{Z}^k$-graded. In particular, $\text{in}_w I$ is generated by Laurent polynomials in the variables $x_{k+1}, \ldots, x_n$ only, so $V(\text{in}_w I)$ equals $T_k^k \times Y$ for some $(d-k)$-dimensional subvariety Y of T_k^{n-k} with defining ideal $J := \text{in}_w I \cap K[x_{k+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$.

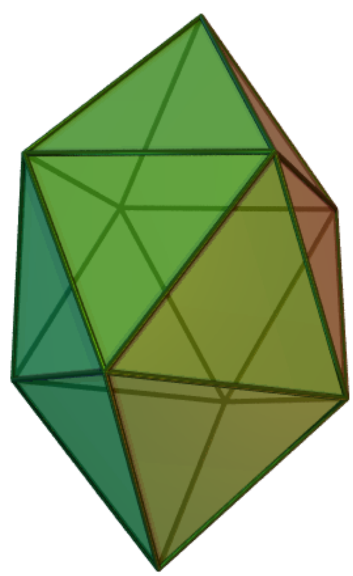
  We will argue that Y is, in fact, 0-dimensional. First let $u \in \mathbb{R}^{n-k} \subseteq \mathbb{R}^n$ be nonzero. By maximality of σ, $w + \epsilon u$ does *not* lie in $\text{trop}(X)$ for any $\epsilon > 0$. Hence $\text{in}_u \text{in}_w I = \langle 1\rangle$. But then already $\text{in}_u \text{in}_w f = 1$ for some $\mathbb{Z}^k$-homogeneous element $f \in I$, which then must lie in J. Hence $\text{in}_u J = \langle 1\rangle$

for all nonzero u. In other words, $\mathrm{trop}(Y) = \{0\}$ (where we think of k with the trivial valuation) and hence Y is finite, hence $d - k = 0$.]

- Sketch of original proof due to Bieri-Groves: we know that, for $X \subseteq T^n$ irreducible of dimension d, $\mathrm{trop}(X)$ is a polyhedral complex of dimension at most d. In the special case where X is a hypersurface, i.e., where $n = d+1$, we know that it is pure of dimension d. In the general case, find a torus homomorphism $\pi : T^n \to T^{d+1}$ such that $Y := \overline{\pi(X)}$ still has dimension d. Then $\mathrm{trop}(\pi)\mathrm{trop}(X) = \mathrm{trop}(Y)$, so in particular $\mathrm{trop}(X)$ must have dimension at least d, hence equal to d. Moreover, suppose that $\mathrm{trop}(X)$ has a maximal cell τ of dimension $e < d$. Pick a point w in the relative interior of that cell. For each cell $\sigma \neq \tau$, the linear span of $w - \sigma$ has dimension at most $d + 1$, so we can choose π such that $\ker \mathrm{trop}(\pi)$ intersects it trivially, and indeed such that this holds for all maximal cells $\sigma \neq \tau$. This means that $\mathrm{trop}(\pi)(w) \notin \mathrm{trop}(\pi)(\sigma)$ for all such σ, so $\mathrm{trop}(\pi)\tau$ is a maximal cell of dimension $\leq e < d$ in the tropicalisation of a hypersurface, a contradiction.

Something about exercises

(1) The first exercise was no problem to anyone.
(2) The second exercise was harder; Arthur, presented his solution.
(3) The last exercise was somewhat tedious, but led to the following beautiful Gröbner complex:



New homework, to be handed in Monday 7 December, 13:00

(1) (In this exercise you may use the programme `gfan_groebnerfan`.) Let A be the skew-symmetric 6×6-matrix

$$A = \begin{bmatrix} 0 & a & b & c & d & e \\ -a & 0 & f & g & h & i \\ -b & -f & 0 & j & k & l \\ -c & -g & -j & 0 & m & n \\ -d & -h & -k & -m & 0 & o \\ -e & -i & -l & -n & -o & 0 \end{bmatrix}.$$

Its determinant is the square of the following polynomial, called the *Pfaffian* of A:

$$f = ehj - dij - egk + cik + dgl - chl + efm - bim + alm - dfn + bhn - akn + cfo - bgo + ajo.$$

Determine:

(a) the lineality space of $\mathrm{trop}(V(f))$ (i.e., the intersection of the lineality spaces of all cones); and

(b) one representative of each orbit of S_6 on the maximal-dimensional cones of $\mathrm{trop}(V(f))$.

(2) Exercise 13 from 3.7.

(3) Consider the rational map from T to T^3 defined by

$$x \mapsto (x - t, x - (t + t^2), x - 1),$$

where $t \in K$ has valuation 1. Let X be the Zariski-closure of the image of this map.

(a) Determine $v(X) \subseteq \mathbb{R}^3$ under the assumption that $v(K^*) = \mathbb{R}$.

(b) Draw the tropical variety of X.

(c) Verify balancing at each of the 0-dimensional cells of $\mathrm{trop}(X)$ (the 1-dimensional cells have multiplicity 1).