TROPICAL GEOMETRY, LECTURE 1

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1. Some tropical arithmetic

- The *tropical semifield* is $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ equipped with $\oplus$ and $\odot$ defined by $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$. Following the convention for classical operations, $\odot$ takes priority over $\oplus$, so $a \oplus b \odot c := a \oplus (b \odot c)$.
- $\oplus$ is commutative, associative, with neutral element ∞; $\odot$ is commutative, associative, with neutral element 0; and $\odot$ distributes over $\oplus$. These properties axiomatise a *semiring*. Here we have a *semifield*: each non-zero element (i.e., each element $a \neq \infty$) has a multiplicative inverse (namely, $-a$).
- but $\oplus$ does not have inverses, so not a ring!
- Arithmetic operations such as scalar times a vector, sum of vectors, matrix products have tropical analogues: replace plus by $\oplus$ and times by $\odot$.
- We will follow [] in often writing classical expressions between double quotation marks to indicate that the operations should be interpreted tropically. So "$1 + 3x^3$" $= 1 \oplus (3 \odot x \odot x \odot x)$.
- Example: draw the "linear span" $\{$"$a(0,0) + b(2,1)$" $\mid a, b \in \mathbb{R}_\infty\} \subseteq \mathbb{R}^2_\infty$.
- Example: Let $D \in [0, \infty]^{n \times n}$ with zeroes on the diagonal. Then "D^{n-1}" records the shortest-path lengths between vertices in the graph where the length of the edge from i to j is d_{ij}.
- Remark: If we consider, instead, the matrix $A = (\epsilon^{d_{ij}})$ where ϵ is a variable, and compute the ordinary n-th power A^n of A, then the lowest exponent of ϵ appearing in its (i, j)-entry is the corresponding entry of "D^n".

2. Tropical univariate polynomials

- Polynomials $f =$ "$\sum_{i=0}^{d} c_i x^i$" (finitely many terms with $c_i \neq \infty$) and $g =$ "$\sum_{j=0}^{e} d_j x^j$" are added as

$$\text{“}f + g\text{”} = f \oplus g = \text{“}\sum_i (c_i + d_i) x^i\text{”}$$

and multiplied as

$$\text{“}fg\text{”} = f \odot g = \text{“}\sum_k (\sum_{i+j=k} c_i d_j) x^k\text{”}.$$

They form a semiring ($\mathbb{R}_\infty$-semialgebra) $\mathbb{R}_\infty[x]$.
- Note that f defines a function $\mathbb{R}_\infty \to \mathbb{R}_\infty$ sending a to $\min_i(c_i + ia)$. This function, being the minimum of linear affine-functions, is concave.
- Example: $f :=$ "$(x + 2)^3$" $=$ "$(0x + 2)^3$" $=$ "$(x^2 + 2x + 4)(x + 2)$" $=$ "$x^3 + 2x^2 + 4x + 6$".

- On the other hand, for each $a \in \mathbb{R}_\infty$, we have $(a \oplus 2)^{\odot 3} = \min\{a + a + a, 2 + 2 + 2\} =$ "$a^3 + 6$". So the polynomials f and $g :=$ "$x^3 + 6$" define the *same function* $\mathbb{R}_\infty \to \mathbb{R}_\infty$. We will mostly be interested only in the function that a polynomial defines.

Proposition 2.1. *Every continuous function $f : \mathbb{R}_\infty \to \mathbb{R}_\infty$ that is concave and linear with integral slopes at all but finitely many points comes from some tropical polynomial in $\mathbb{R}_\infty[x]$. Moreover, there is a unique such polynomial of the form "$c\prod_{i=1}^{d}(x + b_i)$".*

Proof. Let the distinct slopes of f, from left to right, be $d_1, d_2, \ldots, d_k \in \mathbb{Z}$, and let $a_1 < \ldots < a_{k-1} < \infty$ be the points in $\mathbb{R}$ where the slope changes. As f is concave, we have $d_1 > d_2 > \ldots > d_k$. As it extends to all of $\mathbb{R}_\infty$ we must have $d_k \geq 0$. Set $e_i := d_i - d_{i+1}$ for $i = 1, \ldots, k-1$, so that $d := d_1 = e_1 + \ldots + e_{k-1} + d_k$. Then the function f agrees with that determined by the polynomial

$$\text{“}cx^{d_k}\prod_{i=1}^{k-1}(x + a_i)^{e_i}\text{”}$$

for suitable $c \in \mathbb{R}$. For uniqueness, observe that the a_i and c are uniquely determined by the graph of f. □

We will call the a_i the *roots* of (all tropical polynomials representing) f and the e_i their multiplicities. This terminology is justified by a close relation to roots of ordinary polynomials, discussed in the next section.

3. Valued fields

Definition 3.1. Let K be a field. Then a function $v : K \to \mathbb{R}_\infty$ is a *valuation* if $v^{-1}(\infty) = \{0\}$ and $v(a + b) \geq \min\{v(a), v(b)\}$ and $v(ab) = v(a) + v(b)$.

Example 3.2.
- Let $K = \mathbb{C}((t))$, the field of Laurent series $c := \sum_{i \in \mathbb{Z}_{\geq N}} c_i t^i$, where we allow N to be any integer, and set $v(c) := \min\{i \mid c_i \neq 0\}$.
- Let $K = \mathbb{Q}$ and set $v(m/n) :=$the number of factors 3 in m minus the number of factors 3 in n. This is the 3-*adic* valuation. Similarly with other primes p.
- Let K be arbitrary and set $v(c) = 0$ for $c \neq 0$ and $v(0) = \infty$; the *trivial valuation.*

Lemma 3.3. $v(1) = v(-1) = 0$ *and* $v(a) = v(-a)$ *and* $v(a) < v(b) \Rightarrow v(a + b) = v(a)$.

Proof. $v(1) = v(1 \cdot 1) = v(1) + v(1)$ proves the first; and $0 = v(1) = v((-1)^2) = v(-1) + v(-1)$ the second. For the last, note that $v(a + b) > v(a)$ would lead to $v(a) = v((a + b) - b) \geq \min\{v(a + b), v(b)\} > v(a)$. □

Definition 3.4. Given $p \in K[x], p = c_0 + c_1x + \cdots + c_dx^d$, define $\operatorname{trop}(p) :=$ "$v(c_0) + v(c_1)x + \cdots + v(c_d)x^d$" $\in \mathbb{R}_\infty[x]$, the *tropicalisation* of p.

Proposition 3.5 (Newton)**.** *Suppose that p factors completely over K, so that it has roots $a_1, \ldots, a_d$, listed with multiplicities. Then the roots of* $\operatorname{trop}(p)$ *are $v(a_1), \ldots, v(a_d)$, listed with multiplicities.*

This uses the following lemma, which says that trop yields a multiplicative map from polynomials to functions $\mathbb{R}_\infty \to \mathbb{R}_\infty$ (but not to $\mathbb{R}_\infty[x]$).

Lemma 3.6 (Gauss)**.** *For* $p, q \in K[x]$ *the tropical polynomials* $\operatorname{trop}(pq), \operatorname{trop}(p) \odot \operatorname{trop}(q) \in \mathbb{R}_\infty[x]$ *define the same functions* $\mathbb{R}_\infty \to \mathbb{R}_\infty$.

Proof. Write $p = p_0 + \cdots + p_d x^d$ and $q = q_0 + \cdots q_e x^e$. Then, at a fixed $x \in \mathbb{R}_\infty$, the k-th term of $\operatorname{trop}(pq)$ is (in classical notation)

$$v(\sum_{i+j=k} p_i q_j) + kx \geq \min_{i+j=k} v(p_i q_j) + kx = \min_{i+j=k} (v(p_i) + ix) + (v(q_j) + jx),$$

which is at least

$$\min_i (v(p_i) + ix) + \min_j (v(q_j) + jx) = \operatorname{trop}(p)(x) \odot \operatorname{trop}(q)(x).$$

This shows one inequality. For the opposite, we need to show (still, for a fixed $x \in \mathbb{R}_\infty$) that there is a k for which the inequalities are equalities. Take i_0 minimal where $\min_i(v(p_i)+ix)$ is attained and j_0 minimal where $\min_j(v(q_j)+jx)$ is attained and set $k := i_0 + j_0$. For $(i, j) \neq (i_0, j_0)$ with $i + j = k$ we have either $i < i_0$ or $j < j_0$. In either case we have

$$v(p_i q_j) + kx = v(p_i) + ix + v(q_j) + jx > v(p_{i_0}) + i_0 x + v(q_{j_0}) + j_0 x = v(p_{i_0} q_{j_0}) + kx$$

so $v(p_i q_j) > v(p_{i_0} q_{j_0})$. Then, by Lemma 3.3 above, it follows that the k-th term in $\operatorname{trop}(pq)$ equals

$$v(p_{i_0} q_{j_0}) + kx = \operatorname{trop}(p)(x) + \operatorname{trop}(q)(x).$$

□

Proof of Newton's proposition. By Gauss's lemma, the proposition reduces to the case of linear polynomials $p = x - a$. Now $\operatorname{trop}(p) = x \oplus v(a)$, which defines the piecewise linear function that equals x for $x \leq v(a)$ and $v(a)$ for $x \geq v(a)$. □

4. Tropical plane curves

- Let $f =$ "$\sum_{(i,j)\in A} c_{ij} x^i y^j$" $\in \mathbb{R}_\infty[x, y]$ where $A \subseteq \mathbb{Z}^2_{\geq 0}$ is a finite set of exponent pairs and $c_{ij} \neq \infty$ for all $(i, j) \in A$.
- This defines a function $\mathbb{R}^2_\infty \to \mathbb{R}_\infty$ given by

  $$(x, y) \mapsto \min_{(i,j)\in A} (c_{ij} + ix + jy).$$

  This function is continuous, concave, piecewise linear with finitely many linear pieces, where it has integral slopes.
- Like in the univariate case, every function with these properties is determined by some bivariate tropical polynomial. ("Unique factorisation" does not hold, though.)
- *Evaluating* f at a point (x, y) corresponds to solving a linear program, as follows. Let $A' \subseteq \mathbb{R}^3$ be the set of the points (i, j, c_{ij}) with $(i, j) \in A$, so that $f(x, y) = \min_{a\in A'} (x, y, 1) \cdot a$. Since the function $a \mapsto (x, y, 1) \cdot a$ is linear, the minimum remains the same if we replace the domain A' by its convex hull P. The subset of P where the minimum is attained will be either a vertex, or an edge, or a two-dimensional polygon. Since the last coordinate of the linear function is positive, this minimising face will be "visible from below".

- We now come to a notion whose higher-dimensional generalisation will be one of the most important notions in this course:

$$V(f) := \{(x, y) \in \mathbb{R}_\infty \mid f \text{ is either infinite or nonlinear at } (x, y)\} \subseteq \mathbb{R}^2_\infty$$

  is the *tropical curve* defined by f. Let's not worry too much about ∞ at this point, and just look at $(x, y) \in \mathbb{R}$.
- Example: $f =$ "$2 + 0x + 1y$".
- Example: $f =$ "$2 + 1x + 3x^2 + 1y + 0xy + 2y^2$".
- Higher-degree examples are best understood through the linear programming interpretation above: if the minimising face is a vertex, then f is locally linear near (x, y). However, if it is an edge or even a polygon, i.e., if there are at least two terms in the definition of f where the minimum is attained, then f is not locally linear near (x, y).
- The edges of P visible from below correspond to pairs $(i, j) \neq (i', j')$ with the property that there exist $(x, y) \in \mathbb{R}^2$ such that $c_{ij} + ix + jy = c_{i'j'} + i'x + j'y < c_{k,l} + kx + ly$ for all $(k, l) \in A \setminus \{(i, j), (i', j')\}$. The set of such (x, y) forms an open interval in $\mathbb{R}^2$.
- The edges of P seen from below project down into the convex hull of A, which is called the *Newton polygon* of f. They give a subdivision of that polygon into polygons. This subdivision is dual to $V(f)$ in the following sense:
  (1) edges of the subdivision correspond bijectively to line segments of $V(f)$;
  (2) the segment is perpendicular to the edge;
  (3) in this correspondence, boundary edges of the subdivision correspond to infinite rays of $V(f)$;
  (4) polygons in the subdivision correspond to *vertices* in $V(f)$, i.e., points where several edges meet;
  (5) vertices in the subdivision correspond to connected components of the complement of $V(f)$.
- We equip the edges (and rays) of $V(f)$ with positive integral weights, equal to the *lattice length* of the corresponding edge in the dual subdivision. This is the number of integral points on it minus one. [More generally, if ℓ is a finite line segment in $\mathbb{R}^n$ with rational slope, then its lattice length is defined as follows: pick a vector v in $\mathbb{Z}^n$ parallel to ℓ with gcd of its coordinates 1 (i.e., a primitive vector). Then the lattice length of ℓ is the ordinary length of ℓ length divided by the ordinary length $||v||$.]
- Thus $V(f) \cap \mathbb{R}^2$ is a finite, one-dimensional polyhedral complex with rational slopes. Moreover, it satisfies an interesting *balancing property*, as follows. Pick a vertex of $V(f)$, corresponding to a polygon Q in the subdivision. In the directions of each of its line segments, choose a primitive vector, say $v_1, \ldots, v_k \in \mathbb{Z}^2$ in counterclockwise order. Let $m_1, \ldots, m_k$ be the *multiplicities*: the lattice lengths of the corresponding edges of Q. Then $\sum_i m_i v_i = 0$. Indeed, rotating each v_i counterclockwise by 90 degrees, this sum just corresponds to the sum of the vectors along the edges of Q!.