EXTENSION OF SCALARS

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Let V be a vector space over a field $\mathbb{F}$ and let $\mathbb{K} \supseteq \mathbb{F}$ be a field extension. We want to define a vector space $V_{\mathbb{K}}$ together with an $\mathbb{F}$-linear embedding $V \to V_{\mathbb{K}}$ in a natural manner.[1]

The idea is, loosely speaking, that we compute with vectors in V as if they were elements of a vector space over $\mathbb{K}$. To make this idea precise, take a $\mathbb{K}$-vector space H (for "Heel groot", very large) with *basis* consisting of *symbols* $\overline{v}$, one for each $v \in V$ (including 0!). Hence the dimension of H is the cardinality of V, so infinite if the field is infinite and V has positive dimension. Now we introduce relations that identify addition and scalar multiplication in H with those in V whenever possible. More precisely, we define the $\mathbb{K}$-subspace U of H by

$$U := \langle \{\overline{v_1 + v_2} - (\overline{v_1} + \overline{v_2}) \mid v_1, v_2 \in V\} \cup \{\overline{cv} - c(\overline{v}) \mid c \in \mathbb{F}, v \in V\}\rangle_{\mathbb{K}}.$$

Note that it would not make sense to take $c \in \mathbb{K} \setminus \mathbb{F}$ in the last set of spanning elements of U, since multiplication with $c \in \mathbb{K}$ does not make sense in V. We define $V_{\mathbb{K}}$ as the *quotient* H/U. Note that we have a natural map $\beta : V \to V_{\mathbb{K}}$ that sends v to the coset $\overline{v} + U$. We claim that this map is $\mathbb{F}$-linear. Indeed, for $v_1, v_2 \in V$ we have

$$\beta(v_1 + v_2) = \overline{v_1 + v_2} + U = \overline{v_1} + \overline{v_2} + U = \beta(v_1) + \beta(v_2),$$

where we have used the first set of spanning elements of U. Second, for $c \in \mathbb{F}$ and $v \in V$ we have

$$\beta(cv) = \overline{cv} + U = c\overline{v} + U,$$

where we have used the second set of spanning elements of U. Note that we have used all spanning elements of U in this proof. As a consequence, if α is any $\mathbb{K}$-linear map from H to a $\mathbb{K}$-space W such that the map $V \to W$, $v \mapsto \alpha(\overline{v})$ is $\mathbb{F}$-linear, then the kernel of α must *contain* U. This observation is used in the proof of the following theorem.

Theorem 0.1. *For any $\mathbb{F}$-linear map ϕ from V into a $\mathbb{K}$-vector space W there exists a unique $\mathbb{K}$-linear map ψ from $V_{\mathbb{K}}$ into W such that $\phi = \psi \circ \beta$.*

Proof. We are forced to set $\psi(\beta(v)) := \phi(v)$ and since the every element of $V_{\mathbb{K}}$ is a $\mathbb{K}$-linear combination of elements of the form $\beta(v)$ (check this!), the requirement that ψ be $\mathbb{K}$-linear makes ψ unique. To prove existence, we use the property of U above, as follows. There is a unique $\mathbb{K}$-linear map $\psi' : H \to W$ that maps $\overline{v}$ to $\phi(v)$ (recall that the $\overline{v}$ form a basis of H). Now we want to set $\psi(h + U) := \psi'(h)$. To

Date: 10 September 2010.

[1] *Natural* is a loosely mathematical term that you only get a feeling for if you have seen enough examples. It means something like "without needing to make any further choices". For instance, an abstract vector space has no natural basis, but the space $\mathbb{F}^n$ does. An abstract, finite-dimensional vector space V is not in a natural way linearly isomorphic to its dual space V', even though they have the same dimension, but if V comes equipped with a non-degenerate bilinear form, then such a natural isomorphism does exist, etc.

prove that this is well-defined, we need to check that U is contained in the kernel of ψ'. But this follows from the fact that the map $V \to W$ sending v to $\psi'(\overline{v})$ is ϕ and hence $\mathbb{F}$-linear. $\square$

The statement of the theorem is usually referred to as a *universal property* of $V_{\mathbb{K}}$ and $\beta : V \to V_{\mathbb{K}}$.

Exercise 0.2. Use the theorem to prove that if $v_1, \ldots, v_n$ is an $\mathbb{F}$-basis of V, then $\beta(v_1), \ldots, \beta(v_n)$ is a $\mathbb{K}$-basis of $V_{\mathbb{K}}$. Hint: use the basis of V for an $\mathbb{F}$-linear map into $W = \mathbb{K}^n$.

The exercise implies that $\dim_{\mathbb{K}} V_{\mathbb{K}} = \dim_{\mathbb{F}} V$. Often we *identify* V with its image $\beta(V)$ in $V_{\mathbb{K}}$; this is an $\mathbb{F}$-linear subspace of the $\mathbb{K}$-space $V_{\mathbb{K}}$. For instance, $(\mathbb{F}^n)_{\mathbb{K}}$ is identified with $\mathbb{K}^n$, and under this identification the standard basis of $\mathbb{F}^n$ is mapped to the standard basis of $\mathbb{K}^n$.