EXERCISES FOR WEEK 9

JAN DRAISMA

In the lecture on 9 November I discussed bilinear forms, and their orbits (in the symmetric case) over fields $\mathbb{F}$ of characteristic unequal to 2. (In particular, for $\mathbb{F}$ closed under taking square roots, and for $\mathbb{F} = \mathbb{R}$, and for $\mathbb{F}$ finite.)

Exercise 0.1. Let f be an alternating form on a finite-dimensional vector space V.

(1) Prove that there exists a basis $v_1, \ldots, v_n$ of V whose Gram matrix with respect to f is a block-diagonal matrix $\mathrm{diag}(A_1, \ldots, A_k, 0, \ldots, 0)$, where $2k \leq n$, each A_i equals the matrix
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$
and there are $n - 2k$ trailing zeroes (1×1-blocks on the diagonal).

(2) Conclude that the rank of any alternating bilinear form is even.

(3) Conclude that there are exactly $\lfloor n/2 \rfloor + 1$ $\mathrm{GL}(V)$-orbits of alternating bilinear forms.

Exercise 0.2. Give two symmetric bilinear forms on $V = \mathbb{F}_5^2$ that have both rank two but are in different $\mathrm{GL}_2(\mathbb{F}_5)$-orbits.

Date: 10 November 2010.