EXTRA EXERCISE FOR WEEK 4

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Theorem 0.1. *Let W be a finite-dimensional vector space over $\mathbb{K}$, and let $\mu \in \mathcal{L}(W)$ be a nilpotent linear map. Then there exists a basis β of W such that $[\beta^{-1}\mu\beta]$ is a block diagonal matrix with diagonal blocks of the form*

$$J_{x-0,m} := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

where $m \in \mathbb{N}$ is the number of rows (and columns) of this Jordan block.

Note: if a matrix A is of the form in the theorem, with Jordan blocks of sizes $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_l > 0$, then we have

$$m'_j := \dim\ker A^j - \dim\ker A^{j-1} = |\{i \in \{1,\ldots,l\} \mid m_i \geq j\}|.$$

The sequence $m'_1 = l, m'_2, \ldots$ is (weakly) decreasing, and the largest index j for which m'_j is non-zero equals d, the largest size of a Jordan block in A. Below we will find the "last" basis vector for each Jordan block; the other basis vectors for that Jordan block are then obtained by applying the linear map.

Proof. Note that

$$0 \subseteq \ker\mu \subseteq \ker\mu^2 \subseteq \ldots \subseteq \ker\mu^{d-1} \subseteq \ker\mu^d = W$$

where d is minimal with $\mu^d = 0$. Also note that μ maps $\ker\mu^{i+1}$ into $\ker\mu^i$. For $i = 1,\ldots,d$ let U_i denote a vector space complement to $\mu\ker\mu^{i+1} + \ker\mu^{i-1}$ in $\ker\mu^i$. Then μ^j is injective on U_i for all $j = 0,\ldots,i-1$, so that $\mu^j U_i$ has the same dimension as U_i. We claim that in the μ-stable subspace

$$W_i := U_i + \mu U_i + \ldots + \mu^{i-1}U_i$$

of W the sum is in fact direct. Indeed, suppose that

$$u_0 + \mu u_1 + \ldots + \mu^{i-1}u_{i-1} = 0,$$

where all u's are elements of U_i. Applying μ^{i-1} to this yields that u_0 is zero. Applying μ^{i-2} then yields that u_1 is zero, etc. Hence $u_0,\ldots,u_{i-1}$ are all zero; note that we have only used that U_i intersects $\ker\mu^{i-1}$ trivially. After choosing any basis C_i of U_i, we obtaine a basis

$$B_i := \bigcup_{j=0}^{i-1} \mu^j C_i$$

of W_i, and after ordering B_i suitably, we obtain an ordered basis β_i such that $[\beta_i^{-1}\mu|_{W_i}\beta_i]$ consists of $\dim U_i$ Jordan blocks of dimension i. Next we claim that

Date: 23 September 2010.

W is the direct sum of all W_i with $i = 1, \ldots, d$; here we will use that U_i intersects $\operatorname{im}\mu + \ker\mu^{i-1}$ trivially (check this). This implies that $\mu^j U_i$ intersects $\operatorname{im}\mu^{j+1}$ trivially, for all $j = 0, \ldots, i-1$: if $\mu^{j+1}w = \mu^j u$ with $u \in U_i$ and $w \in W$, then $\mu^j(\mu w - u) = 0$, and hence in particular $u - \mu w \in \ker\mu^{i-1}$, so that $u \in \operatorname{im}\mu + \ker\mu^{i-1}$, hence $u = 0$. Now suppose that $\sum_{i=1}^{d}\sum_{j=0}^{i-1}\mu^j u_{ij} = 0$, where $u_{ij} \in U_i$. Applying μ^{d-1} we find that $\mu^{d-1}u_{d0} = 0$, so $u_{d0} = 0$. Then applying μ^{d-2} we find that $\mu^{d-1}u_{d1} + \mu^{d-2}u_{d-1,0} = 0$. By the above, $u_{d-1,0} = 0$ and hence also $u_{d,1} = 0$. Then applying μ^{d-3} we find, consecutively, $u_{d-2,0} = 0$ and $u_{d-1,1} = 0$ and $u_{d,2} = 0$, etc. This means that the sum in $W_1 + \ldots + W_d$ is direct. Finally, this sum is also all of W, as you are asked to prove below. □

Exercise 0.2. (1) Prove that W is spanned by the W_i.
(2) Determine where the following alternative "proof" of the theorem is incorrect, by giving a counter-example.

Fake proof. Proceed by induction on the dimension of W. First, for W of dimension 0 nothing needs to be done. Next, suppose that the theorem is true for all W' of dimension $< n$ and let W be of dimension $n > 1$. Choose any non-zero vector $u \in W$ and let e be minimal with $\mu^e u = 0$. Then the vectors $u, \mu u, \ldots, \mu^{e-1}u$ are linearly independent. They form a basis of a μ-stable subspace U of W, relative to which μ has as matrix a single Jordan block of size e. Now let W' be a μ-stable complement of U in W. Since W' has dimension $n - e < n$, it has a suitable basis β' by the induction hypothesis. The matrix of U with respect to the concatenation of the basis β' and the basis above for U' has the required property. □