EXERCISES FOR WEEK 14

JAN DRAISMA

In the lecture I discussed two prototypical examples of the linear algebra method in combinatorics. First, the (uniform and non-uniform) RW-theorem. See the handouts by Babai and Frankl from weeks ago, Chapter 5. Second, equiangular lines in $\mathbb{R}^n$ (as many lines as possible through 0 in $\mathbb{R}^n$ that pairwise make the same angle). Here is the complex analogue:

Exercise 0.1. Let $v_1, \ldots, v_k$ be column vectors in $\mathbb{C}^n$. Write v_i^* for the row vector obtained by transposing v_i and taking the complex conjugates of its entries. Assume that first, $(||v_i||^2 =) v_i^* v_i = 1$ for all i, i.e., the vectors have norm 1; and second, the absolute value $|v_i^* v_j|$ is a constant $c \neq 1$ independent of i and j (this is the complex analogue of "having pairwise the same angle").

(1) Prove that $c < 1$.
(2) Prove that $k \leq n^2$ by adapting the proof in the real case. Hint: prove that the Hermitian matrices $A_i := v_i v_i^*$ are linearly independent by computing their "Gramm matrix" relative to the Hermitian form $f(A, B) := \mathrm{tr}(B^* A)$.