EXERCISES FOR WEEK 10

JAN DRAISMA

In the lecture I discussed σ-sesquilinear forms, and in particular σ-Hermitian forms. For some pairs of a field $\mathbb{F}$ and an automorphism σ of order two, I described their orbits.

Exercise 0.1. Let V be an n-dimensional vector space over $\mathbb{C}$ and let σ denote complex conjugation.

(1) Let f and h be σ-Hermitian forms on V. Prove that if f and h are in the same $\mathrm{GL}(V)$-orbit (that is, there exists an invertible $g \in \mathcal{L}(V)$ such that $f(g^{-1}v, g^{-1}w) = h(v, w)$ for all v, w), then their ranks must be the same.

(2) Let f and h be σ-Hermitian forms on V with the same radical $U = \mathrm{Rad}(f) = \mathrm{Rad}(h)$. Then they induce σ-Hermitian forms on the quotient space V/U by $\tilde{f}(v+U, w+U) = f(v, w)$, and similarly for h. Prove that f and h are in the same $\mathrm{GL}(V)$-orbit if and only if $\tilde{f}$ and $\tilde{h}$ are in the same $\mathrm{GL}(V/U)$-orbit.

We have seen (in the exercise class) that for every σ-Hermitian form f on V there exists a basis $v_1, \ldots, v_n$ of V whose Gram matrix is diagonal with (consecutively, from top left to bottom right) p ones, q minus-ones, and $n - p - q$ zeroes on the diagonal.

(3) Let f be a σ-Hermitian form, and choose a basis and p and q as above. Prove that for every non-zero v in the span of $v_1, \ldots, v_p$ we have $f(v, v) > 0$.

(4) Prove that for every subspace W of V of dimension strictly larger than p there exists a non-zero $w \in W$ with $f(w, w) \leq 0$.

(5) Conclude that p is determined uniquely by f, and that there are exactly $\frac{1}{2}(n+1)(n+2)$ $\mathrm{GL}(V)$-orbits of σ-Hermitian forms on V.