Tentamen 2DN12 (Lineaire Algebra)
28-6-2010

1. Vegen:

$$\left[\begin{array}{ccc|c} \lambda & 1 & 1 & 3 \\ 1+\lambda & 2 & 2 & 6 \\ 0 & 3 & 3+\lambda & 7 \end{array}\right] \sim \overset{(**)}{\left[\begin{array}{ccc|c} \lambda & 1 & 1 & 3 \\ 1-\lambda & 0 & 0 & 0 \\ 0 & 3 & 3+\lambda & 7 \end{array}\right]}$$

$$\overset{\lambda \neq 1:}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & \lambda & -2 \end{array}\right] (*) \overset{\lambda \neq 0}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3+\frac{2}{\lambda} \\ 0 & 0 & 1 & -\frac{2}{\lambda} \end{array}\right]$$

enige opl als $\lambda \notin \{1, 0\}$: $\boxed{x = 0,\ y = 3 + \frac{2}{\lambda},\ z = -\frac{2}{\lambda}}$

als $\lambda = 0$: $\boxed{\text{geen opl. want } (*) \text{ strijdig}}$

als $\lambda = 1$:

$$(**) \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 7 \end{array}\right]$$

opl. verzameling $= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \right\rangle$

ofwel: $\boxed{(x, y, z) = (1+s,\ 1-4s,\ 1+3s) \text{ voor } s \in \mathbb{R}.}$

2. a. Gram-Schmidt:

$$\underline{v}_1 = (1,2,2,0) \qquad \underline{v}_2 = (3,2,1,2)$$

$$\underline{u}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|} = \frac{1}{\sqrt{9}} \cdot (1,2,2,0) = \boxed{\frac{1}{3}(1,2,2,0)}$$

$$\underline{u}_2 = \frac{\underline{v}_2 - \frac{\underline{v}_2 \bullet \underline{v}_1}{\underline{v}_1 \bullet \underline{v}_1} \cdot \underline{v}_1}{\| \quad \cdot \quad \|}$$

$$= \frac{(3,2,1,2) - \frac{9}{9} \cdot (1,2,2,0)}{\| \quad \cdot \quad \|}$$

$$= \frac{(2,0,-1,2)}{\sqrt{9}} = \boxed{\frac{1}{3}(2,0,-1,2)}$$

b. $$\left((2,1,1,0) \bullet \underline{u}_1\right)\underline{u}_1 + \left((2,1,1,0) \bullet \underline{u}_2\right)\underline{u}_2$$

$$= \frac{2}{3}(1,2,2,0) + \frac{1}{3} \cdot (2,0,-1,2)$$

$$= \boxed{\left(\frac{4}{3}, \frac{4}{3}, 1, \frac{2}{3}\right)}$$

~~a. ⊥ projectie van (1,1,1,0) op U is (als in b):~~

$$\cancel{\left(\frac{4}{3}, \frac{4}{3}, 1, \frac{2}{3}\right) + \frac{1}{9}(4,2,2,0) - \frac{2}{9}(2,0,-1,2)}$$

$$\cancel{=}$$

c. $\perp$ proj. van $(1,1,1,0)$ op U is

$\frac{5}{9}\cdot(1,2,2,0) + \frac{1}{9}(2,0,-1,2)$

$= \frac{1}{9}(7, 10, 9, 2)$

verschil : $(1,1,1,0) - \frac{1}{9}(7,10,9,2)$

$= \frac{1}{9}(2, -1, 0, -2)$

met lengte : $\frac{1}{9}\cdot\sqrt{9} = \boxed{1/3}$.

3.c B: standaardbasis van $\mathbb{R}^3$

C: " " $\mathbb{R}^4$

$B' = \{(1,1,1), (1,1,-1), (1,2,1)\}$

$$[A]_{C,B'} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ 3 & 3 & 6 \\ 4 & 3 & 7 \end{bmatrix}$$

$$[A]_{C,B} = [A]_{C,B'} \cdot \underbrace{P_{B',B}}_{\text{overgangsmatrix van } B \text{ naar } B'}$$

* Eerst $P_{B'B}$ bepalen als $P_{B,B'}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$\uparrow$

$P_{B'B}$

dus $[\mathcal{A}]_{C,B} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ 3 & 3 & 6 \\ 4 & 3 & 7 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}$

$$\boxed{= \begin{bmatrix} -1 & 3 & -1 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \\ \frac{1}{2} & 3 & \frac{1}{2} \end{bmatrix}}$$

a. $\dim \mathcal{N} = 1$, basis : $(1,0,-1)$.

b. $\dim R = 3 - 1 = 2$, basis :

$(1,2,3,4)$, $(3,0,3,3)$.

4. $\underline{x}' = A\underline{x} + \underline{b}$ met $A = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

en $\underline{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a. A : bepaal eigenruimten:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -2 & -2 \\ 1 & \lambda & 0 \\ 1 & 0 & \lambda \end{bmatrix}$$

$$= \lambda^3 + 2\lambda + 2\lambda = \lambda(\lambda^2+4).$$

$$E_0 = \left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle.$$

$$E_{2i} = \ker \begin{bmatrix} 2i & -2 & -2 \\ 1 & 2i & 0 \\ 1 & 0 & 2i \end{bmatrix}$$

$$= \left\langle \begin{bmatrix} 2i \\ -1 \\ -1 \end{bmatrix} \right\rangle \qquad E_{-2i} = \left\langle \begin{bmatrix} -2i \\ -1 \\ -1 \end{bmatrix} \right\rangle$$

opl. ruimte homogene stelsel $\underline{x}' = A\underline{x}$:

$$\left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{0t}, \operatorname{Re}\left(\begin{bmatrix} 2i \\ -1 \\ -1 \end{bmatrix} e^{2it} \right), \operatorname{Im}\left(\begin{bmatrix} 2i \\ -1 \\ -1 \end{bmatrix} e^{2it} \right) \right\rangle$$

$$= \boxed{\left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2\sin 2t \\ -\cos 2t \\ -\cos 2t \end{bmatrix}, \begin{bmatrix} 2\cos 2t \\ -\sin 2t \\ -\sin 2t \end{bmatrix} \right\rangle \quad (*)}$$

particuliere opl.:

probeer eerst $\underline{x}_p = \underline{u} \cdot e^{0t}$: invullen:

$$0 = A\,\underline{u} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

dus $\underline{u} = \begin{bmatrix} +1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$ voldoet.

algemene opl.: $\underline{x} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + \underline{x}_h$ met $\underline{x}_h \in (*)$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \underline{0}$, dan staat er

$$\underline{0} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ dus}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + \underbrace{\left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle}_{\ker A}}$$

5. a. (1) $A(p+q) = (x-1)\cdot(p+q)'(x)$

$= (x-1)\cdot(p'(x) + q'(x))$

$= (x-1)\cdot p'(x) + (x-1)q'(x)$

$= Ap + Aq.$

(2) $A(cp) = (x-1)\cdot(cp)'(x)$

$= c\cdot(x-1)\cdot p'(x)$

$= c\cdot(Ap).$

b. $$\begin{array}{c|ccc} & 1 & x & x^2 \\ \hline 1 & & & \\ x & & \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} & \\ x^2 & & & \end{array} = [A]_B$$ met $B = \{1, x, x^2\}$.

c. $Ap = \lambda p$ betekent: $(x-1)\,p'(x) = \lambda p(x)$

dit geldt i.i.g. $\boxed{\text{voor } p = 1,\ (x-1),\ (x-1)^2 \text{ met } \lambda = 0,\ 1,\ 2.}$

omdat we al 3 eigenwaarden hebben, zijn er geen andere.

d. $x^2 - x = (x-1)^2 + (x-1).$

dus $A^{100}(x^2-x) = A^{100}(x-1)^2 + A^{100}(x-1)$

$= 2^{100}\cdot(x-1)^2 + 1^{100}\cdot(x-1)$

$= 2^{100}\cdot x^2 - (2^{101}-1)x + (2^{100}-1).$

Voor alle $\underline{v} \in \mathbb{R}^n$ geldt:

6\. a. $\underline{v} \cdot A\underline{v} = (A\underline{v})^T \underline{v} = \underline{v}^T A^T \underline{v}$

$$= -\underline{v}^T A \underline{v} = -\underline{v} \cdot (A\underline{v})$$

dus $\underline{v} \cdot (A\underline{v}) = -\underline{v} \cdot (A\underline{v})$

en daarom $\underline{v} \cdot (A\underline{v}) = 0$.

b\. $a_{ij} = \underline{e}_i^T A \underline{e}_j$ met $\underline{e}_1, \ldots, \underline{e}_n$ de standaard-basis.

Er geldt:

$$0 = (\underline{e}_i + \underline{e}_j)^T A (\underline{e}_i + \underline{e}_j) = \underline{e}_i^T A \underline{e}_i + \underline{e}_j^T A \underline{e}_j + \underline{e}_i^T A \underline{e}_j + \underline{e}_j^T A \underline{e}_i$$

$$= 0 + 0 + a_{ij} + a_{ji}$$

~~dus~~

dus $a_{ij} = -a_{ji}$ en $A^T = -A$.