Tentamen 2DN02 (Lin. Afb.)
28-6.

1. a. $P = \frac{1}{\sqrt{13}} \cdot \begin{bmatrix} 3 & -2 \\ +2 & 3 \end{bmatrix}$ $\lambda = 52$ $\mu = -13$

(zie JUNI 2008)

b. $\pm \frac{1}{\sqrt{13}} \left(\frac{3}{2}, 1\right)$ (JUNI 2008)

2. Als $A(a_1, a_2, a_3, \ldots) = \lambda(a_1, a_2, a_3, \ldots)$

$\parallel$

$(a_3, a_4, \ldots)$, dan

$a_3 = \lambda a_1$
$a_4 = \lambda a_2$
$a_5 = \lambda a_3 = \lambda^2 a_1$
$a_6 = \lambda a_4 = \lambda^2 a_2$
$\vdots$

dus $(a_1, a_2, a_3, \ldots)$

$\in \langle (1, 0, \lambda, 0, \lambda^2, 0, \lambda^3, \ldots),$
$(0, 1, 0, \lambda, 0, \lambda^2, 0, \ldots) \rangle =: V_\lambda$

Omgekeerd is elke vector ~~xx~~ $\underline{v} \in V_\lambda$ een opl. van $A\underline{v} = \lambda \underline{v}$.

Dus elke $\lambda \in \mathbb{R}$ is ew., met $E_\lambda = V_\lambda$.

3.c B: standaardbasis van $\mathbb{R}^3$

C: " " $\mathbb{R}^4$

$B' = \{(1,1,1), (1,1,-1), (1,2,1)\}$

$$[A]_{C,B'} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ 3 & 3 & 6 \\ 4 & 3 & 7 \end{bmatrix}$$

$$[A]_{C,B} = [A]_{C,B'} \cdot \underbrace{P_{B',B}}_{\text{overgangsmatrix van } B \text{ naar } B'}$$

* Eerst $P_{B'B}$ bepalen als $P^{-1}_{B,B'}$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right]$$

$\uparrow$

$P_{B'B}$

dus $[\mathcal{A}]_{C,B} = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 2 \\ 3 & 3 & 6 \\ 4 & 3 & 7 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}$

$$\boxed{= \begin{bmatrix} -1 & 3 & -1 \\ 1 & 0 & 1 \\ 0 & 3 & 0 \\ \frac{1}{2} & 3 & \frac{1}{2} \end{bmatrix}}$$

a. $\dim N = 1$, basis: $(1,0,-1)$

b. $\dim R = 3-1 = 2$, basis:

$(1,2,3,4)$, $(3,0,3,3)$.

4. $\underline{x}' = A\underline{x} + \underline{b}$ met $A = \begin{bmatrix} 0 & 2 & 2 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

en $\underline{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a. A ~~[illegible]~~ : bepaal eigenruimten:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -2 & -2 \\ 1 & \lambda & 0 \\ 1 & 0 & \lambda \end{bmatrix}$$

$$= \lambda^3 + 2\lambda + 2\lambda = \lambda(\lambda^2+4).$$

$$E_0 = \left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\rangle.$$

$$E_{2i} = \ker \begin{bmatrix} 2i & -2 & -2 \\ 1 & 2i & 0 \\ 1 & 0 & 2i \end{bmatrix}$$

$$= \left\langle \begin{bmatrix} 2i \\ \cancel{}\ -1 \\ -1 \end{bmatrix} \right\rangle \qquad E_{-2i} = \left\langle \begin{bmatrix} -2i \\ -1 \\ -1 \end{bmatrix} \right\rangle$$

opl. ruimte homogene stelsel $\underline{x}' = A\underline{x}$:

$$\left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{0t}, \operatorname{Re}\left(\begin{bmatrix} 2i \\ -1 \\ -1 \end{bmatrix} e^{2it} \right), \operatorname{Im}\left(\begin{bmatrix} 2i \\ -1 \\ -1 \end{bmatrix} e^{2it} \right) \right\rangle$$

$$= \boxed{\left\langle \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2\sin 2t \\ -\cos 2t \\ -\cos 2t \end{bmatrix}, \begin{bmatrix} 2\cos 2t \\ -\sin 2t \\ -\sin 2t \end{bmatrix} \right\rangle.}$$

(*)

particuliere opl:

probeer eerst $\underline{x}_p = \underline{u} \cdot e^{0t}$: invullen:

$$0 = A\,\underline{u} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

dus $\underline{u} = \begin{bmatrix} +1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$ ~~[illegible]~~ voldoet.

$\boxed{\text{algemene opl: } \underline{x} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + \underline{x}_h \text{ met } \underline{x}_h \in (*)}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \underline{0}$, dan staat er

$$\underline{0} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ dus}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + \Big\langle \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\text{ker } A} \Big\rangle}$$

5. a. (1) $A(p+q) = (x-1)\cdot(p+q)'(x)$

$= (x-1)\cdot(p'(x) + q'(x))$

$= (x-1)\cdot p'(x) + (x-1)q'(x)$

$= Ap + Aq.$

(2) $A(cp) = (x-1)\cdot(cp)'(x)$

$= c\cdot(x-1)\cdot p'(x)$

$= c\cdot(Ap).$

b.
$$\begin{array}{c|c} & \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{array}{c} 1 \\ x \\ x^2 \end{array} & \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix} \end{array} = [A]_B$$
met $B = \{1, x, x^2\}$.

c. $Ap = \lambda p$ betekent: $(x-1)p'(x) = \lambda p(x)$
dit geldt i.i.g. $\boxed{\text{voor } p = 1, (x-1), (x-1)^2 \text{ met } \lambda = 0, 1, 2.}$

omdat we al 3 eigenwaarden hebben, zijn er geen andere.

d. $x^2 - x = (x-1)^2 + (x-1).$

dus $A^{100}(x^2-x) = A^{100}(x-1)^2 + A^{100}(x-1)$

$= 2^{100}\cdot(x-1)^2 + 1^{100}\cdot(x-1)$

$= 2^{100}\cdot x^2 - (2^{101}-1)x + (2^{100}-1).$

Voor alle $\underline{v} \in \mathbb{R}^n$ geldt:

6. a. $\underline{v} \cdot A\underline{v} = (A\underline{v})^T \underline{v} = \underline{v}^T A^T \underline{v}$

$$= -\underline{v}^T A \underline{v}$$

$$= -\underline{v} \cdot (A\underline{v})$$

dus $\underline{v} \cdot (A\underline{v}) = -\underline{v} \cdot (A\underline{v})$

en daarom $\underline{v} \cdot (A\underline{v}) = 0$.

b. $a_{ij} = \underline{e}_i^T A \underline{e}_j$ met $\underline{e}_1, \ldots, \underline{e}_n$ de standaard-basis.

Er geldt:

$$0 = (\underline{e}_i + \underline{e}_j)^T A (\underline{e}_i + \underline{e}_j) = \underline{e}_i^T A \underline{e}_i + \underline{e}_j^T A \underline{e}_j + \underline{e}_i^T A \underline{e}_j + \underline{e}_j^T A \underline{e}_i$$

$$= 0 + 0 + a_{ij} + a_{ji}$$

dus $a_{ij} = -a_{ji}$ en $A^T = -A$.