Invariant theory with applications
assignment week 10

November 18, 2009

Exercise 1

Let G be a closed subgroup of $\mathrm{GL}_n(\mathcal{C})$ and suppose that W is a finite-dimensional G-submodule of $\mathcal{C}[\mathrm{GL}_n(\mathcal{C})]$. Show that for the integer k larger enough, $W' := \det(x)^k W \subseteq \mathcal{C}[(x_{ij})_{ij}]$, and that W is completely reducible if and only if W' is.

Exercise 2

Let $V = \mathcal{C}^2$ be the standard $\mathrm{SL}_2(\mathcal{C})$ module. Show that for every positive integer d, we have the following decomposition of the $\mathrm{SL}_2(\mathcal{C})$ module $S^d(S^2(V))$:

$$S^d(S^2(V)) \cong \bigoplus_{k=0}^{\lfloor d/2 \rfloor} S^{2d-4k}(V). \qquad (1)$$

Hint: Let x, y be a basis of V and $X = x^2, Y = y^2, Z = xy$ a basis of $S^2(V)$. Show that the kernel of the homomorphism $\phi : S^d(S^2(V)) \to S^{2d}(V)$ defined by $\phi(X^a Y^b Z^c) := (x^2)^a (y^2)^b (xy)^c$, is isomorphic to $S^{d-2}(S^2(V))$.