(2) $\alpha(gh, x) = \alpha(g, \alpha(h, x))$ for all $x \in X$ and $g, h \in G$.

Usually, the letter α is omitted, and we write gx or $g \cdot x$ instead of $\alpha(g, x)$. Recall that the map sending g to $\alpha(g, .)$, which itself is a function $X \to X$, is a homomorphism from G into the group $\operatorname{Sym} X$ of all bijections from X into itself. In an algebra course, you have already encountered several group actions:

EXAMPLE 3.2. (1) The automorphism group of a graph acts on the vertices of the graph (in such a way that connected vertices are mapped to connected vertices, and non-connected vertices to non-connected vertices).
(2) A group acts on itself by conjugation.
(3) A group acts on itself by left multiplication.
(4) More generally, if H is a subgroup of G, then G acts by left multiplication on the set G/H of left cosets of H.

DEFINITION 3.3 (Stabiliser, orbit, homogeneous space, group quotient). For given $x \in X$, we call

$$G_x := \{g \in G \mid gx = x\}$$

the *stabiliser* of x and

$$Gx := \{gx \mid g \in G\}$$

the *orbit* of x. If $X = Gx$ for some (and hence all) $x \in X$, then G is said to act *transitively* on X, and X is called a *homogeneous* space under the action of G.

If H is a subgroup of G, then G/H is by definition the set of left cosets gH of H in G. This is in general only a set (though if H is a *normal subgroup*, then it carries a natural group structure). Recall the following well-known lemma.

LEMMA 3.4. *Let G be a group acting on a set X, and let $x \in X$. Then the map $G \to X$, $g \mapsto gx$ factorises through $G \to G/G_x$ and a unique map $\psi : G/G_x \to X$, which is a bijection onto the orbit Gx.*

PROOF. Write $H := G_x$. That ψ is unique follows from the surjectivity of $G \to G/H$—indeed, we must have $\psi(gH) = gx$ for all $g \in G$. Now we claim that this defines ψ unambiguously. Well, suppose that $gH = g'H$. Then we have $g' = gh$ for some $h \in H$, so that $g'x = ghx = gx$ as h stabilises x. So, indeed, ψ is well-defined. Now ψ certainly maps G/H surjectively onto the orbit of x, so that we need only verify that ψ is injective. Well, if $\psi(gH) = \psi(g'H)$, then $gx = g'x$, so that $g^{-1}g' \in G_x = H$ and hence $gH = g'H$. □

EXERCISE 3.5. Consider the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (x : y) = (ax + by : cx + dy),$$

or in the affine chart consisting of points $(z : 1)$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d},$$

so-called *Möbius transformations*. This action gives a map $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{Sym}(\mathbb{P}^1(\mathbb{C}))$.

(1) What is the *kernel* of this action? In other words, which matrices g fix all points in $\mathbb{P}^1(\mathbb{C})$?
(2) What is the stabiliser of 0 in GL_n?

(3) Show that the image of ρ is generated by the transformations $z \mapsto az$, $z \mapsto z+b$, $z \mapsto 1/z$.
(4) Show that every element of $\mathrm{GL}_2(\mathbb{C})$ maps (ordinary) circles and lines in $\mathbb{C}$ to circles and lines in $\mathbb{C}$. Hint: show that $z \mapsto 1/z$ permutes circles that do not pass through 0, permutes lines through 0, and maps circles through 0 to lines not through 0 and vice versa.

4. Homogeneity of projective spaces

We will now show that $\mathbb{P}V$ is homogeneous under a suitable group action.

Definition 4.1. We write $\mathrm{GL}(V)$ for the set of all invertible linear maps $V \to V$ and $\mathrm{GL}_n(K)$ for the group of invertible $n \times n$-matrices with entries in K.

The group $\mathrm{GL}(V)$ acts on the set of $(m-1)$-dimensional projective subspaces of $\mathbb{P}(V)$ as follows: if W is an m-dimensional vector subspace of V and $g \in \mathrm{GL}(V)$, then $\{gw \mid w \in W\}$ is also m-dimensional (as g is invertible), and we denote this space by gW; this gives the action of g. In particular, if we take $m = 1$, this gives an action of G on projective points. Note that relative to a basis of V we can identify $\mathrm{GL}(V)$ with GL_n, and then the action is just by matrix-vector multiplication on homogeneous coordinates. Here is a trivial, but very useful lemma.

Lemma 4.2. *Let $g \in \mathrm{GL}(V)$ and let P, Q be projective subspaces of $\mathbb{P}V$. Then P and Q are incident if and only if gP, gQ are incident.*

(In other words, the group $\mathrm{GL}(V)$ acts by automorphisms on the graph whose vertices are projective subspaces of $\mathbb{P}V$ and whose edges represent incidence.)

Lemma 4.3. *The homomorphism $\mathrm{GL}(V) \to \mathrm{Sym}(\mathbb{P}V)$ given by the action of $\mathrm{GL}(V)$ on (the points of) $\mathbb{P}V$ has as kernel precisely the scalar multiples of the identity map.*

Proof. An element $g \in \mathrm{GL}(V)$ lies in the kernel if and only if it maps every one-dimensional subspace of V into itself. The scalar multiples of I certainly do this. Conversely, let g do this, and let $v \in V \setminus \{0\}$ and $\lambda \in K^*$ with $gv = \lambda v$. We claim that $g = \lambda I$. Indeed, for any $w \notin Kv$ we have $g(v+w) = \lambda v + \mu w$ for some $\mu \in K$, and the latter expression in turn is a scalar multiple of $v + w$. But then $\mu = \lambda$ by linear independence of v and w. □

Definition 4.4. The group $\mathrm{GL}(V)/\{cI \mid c \in K^*\}$ is called the *projective linear group* and denoted $\mathrm{PGL}(V)$. If $V = K^n$, then this group is also denoted $\mathrm{PGL}_n(K)$.

Proposition 4.5. *Still writing $n = \dim V$, the group $\mathrm{GL}(V)$ acts transitively on the set of ordered $(n+1)$-tuples of projective points with the property that each n-tuple of them is projectively independent.*

Proof. Let $(p_1, \ldots, p_{n+1})$ and $(q_1, \ldots, q_{n+1})$ be two such tuples of points. For $i = 1, \ldots, n+1$ let v_i be a non-zero vector in the one-dimensional subspaces p_i. Then $v_1, \ldots, v_n$ is a basis of V, hence we can write $v_{n+1} = \sum_i c_i v_i$. The condition on the p_i implies that all c_i are non-zero, hence by replacing $v_1, \ldots, v_n$ by the scalar multiples $c_1 v_1, \ldots, c_n v_n$ (which represent the same points $p_1, \ldots, p_n$), we may assume that all c_i are 1. Similarly choose $w_1, \ldots, w_n, w_{n+1}$ representing $q_1, \ldots, q_{n+1}$ and satisfying $w_{n+1} = \sum_{i=1}^n w_i$. Now since $v_1, \ldots, v_n$ are a basis of