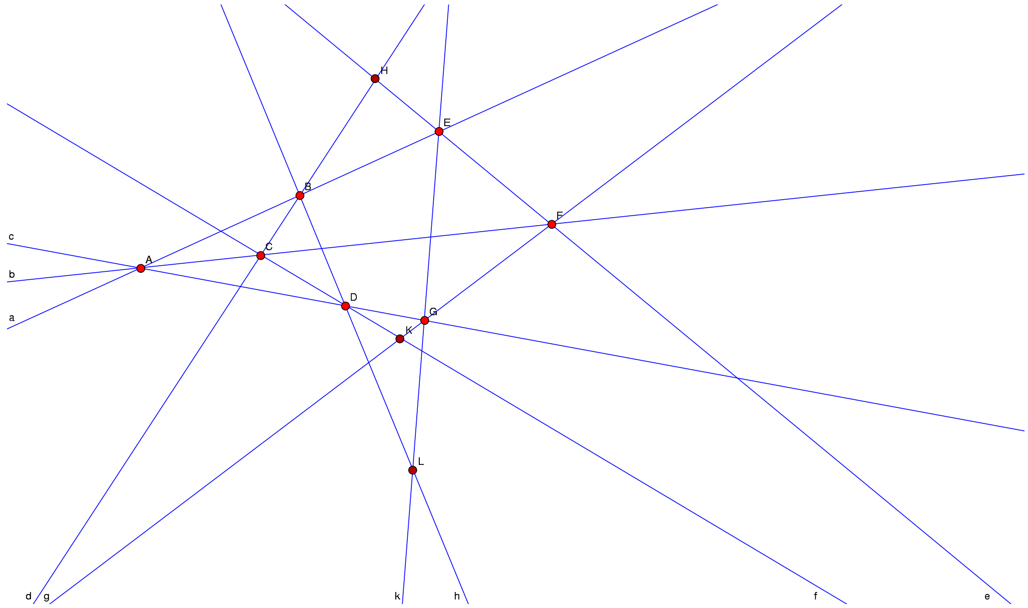

FIGURE 1. Desargues' configuration

LEMMA 2.4. *ϕ and ψ are (well-defined) bijections between $\mathcal{P}$ and $\tilde{P}$ and each other's inverses. Moreover, ϕ maps lines in $\mathcal{L}$ bijectively to lines in $\tilde{L}$ and vice versa.*

EXERCISE 2.5. Prove this lemma.

One of the advantages of the definition of $\mathbb{P}^2(\mathbb{R})$ is that all points look alike. Indeed, the following exercise shows that you could take *any* line in $\mathcal{L}$ to be the line at infinity.

EXERCISE 2.6. Let l be any line in $\mathbb{P}^2(\mathbb{R})$. Show that there are ϕ and ψ with the properties of the lemma for which $\phi(l) = l_\infty$.

3. Some classical geometry

Here is an old theorem by Desargues.

THEOREM 3.1. *In Figure 1, the points H, K, L lie on a line.*

This figure lives in the real projective plane, but by Exercise 2.6 we may assume that all the action takes place in good old $\mathbb{R}^2$.

PROOF. Consider the projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ onto the first two coordinates. A *lift* of $P \in \mathbb{R}^2$ is a point $P' \in \mathbb{R}^3$ with $\pi(P') = P$. For each of the points A, B, C, D choose lifts A', B', C', D' that do not lie on a plane (this is possible). This determines unique lifts $E', \ldots, L'$ of $E, \ldots, L$ by the requirement that $A'B'E'$ etc. are lines in $\mathbb{R}^3$. Now $B'C'D'$ and $E'F'G'$ are distinct planes (otherwise C' would lie on the plane $B'D'E' = A'B'D'$), and their intersection contains each of the points H', K', L'. Hence these lie on a line, and so do their projections H, K, L. $\square$

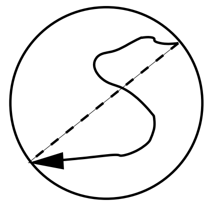

FIGURE 2. A closed path in $\mathcal{P}$.

> **Philosophy.** It is crucial in this proof that the configuration can be seen as a configuration in a higher-dimensional space. We might later encounter geometries satisfying the basic axioms for projective planes, in which Desargues' theorem does *not* hold. Indeed, Desargues' theorem, stated as an axiom, turns out to be *exactly* the extra condition needed for a projective plane to be coordinatisable!

EXERCISE 3.2. Consider the graph with 10 vertices $A, B, \ldots, H, K, L$ and 10, where the vertices are connected by an edge if they do *not* lie on any of the lines drawn in Figure 1, nor on the line H, K, L whose existence is stated by Desargues theorem. Do you recognise this graph? What is its automorphism group?

4. Visualising $\mathbb{P}^2(\mathbb{R})$ and some strange consequences

In this section I want to give you a grasp on the real projective plane, and show how different it is from the Euclidean plane!

Let $S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ be the 2-sphere in $\mathbb{R}^3$. Let π be the map from S^2 to $\mathcal{P}$ sending a point v to the projective point $\mathbb{R}v \in \mathcal{P}$. Note that π is 2-to-1: every point p in $\mathcal{P}$ corresponds to a 1-dimensional subspace of $\mathbb{R}^3$, which intersects S^2 in exactly 2 points—both of whose images under π are p. Similarly, every line l in $\mathbb{P}^2(\mathbb{R})$ corresponds to a 2-dimensional subspace V of $\mathbb{R}^3$, hence to a circle on S^2 centered at the origin—usually called *great circles*—and this great circle is mapped by π onto l. We often only draw the upper half sphere, seen from above, remembering that opposite points on the bordering circle correspond to the *same* point.

Forget about the lines for a moment, and just consider the point set $\mathcal{P}$. This set comes with a natural *topology*. If you do not know what a topology is, then don't worry too much (but do take the next opportunity in your mathematics career to learn it!): it is a structure on $\mathcal{P}$ which says what maps to and from $\mathcal{P}$ (from and to other topological spaces) are *continuous*. (If you do know what a topological space is: a subset of $\mathcal{P}$ is called open if its preimage under π in S^2 is open.) For instance, Figure 2 depicts the image of a continuous closed path, i.e., a continuous map $c : [0, 1] \to \mathcal{P}$ with $c(0) = c(1)$—recall that opposite points on the circle are to be identified!

4.1. Compactness. By definition of the topology on $\mathcal{P}$, the surjective map $\pi : S^2 \to \mathcal{P}$ is continuous. Since S^2 is compact, so is $\mathcal{P}$. Note that this is already quite a difference with $\mathbb{R}^2$: by adding the line at infinity, we have embedded $\mathbb{R}^2$ into a nice compact space.

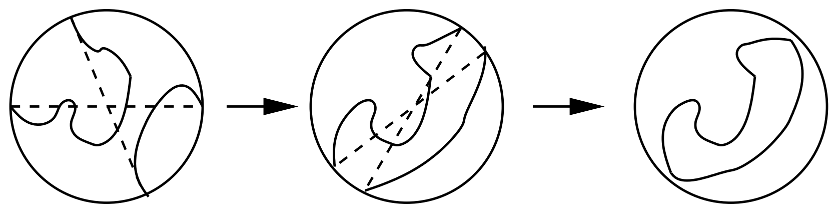

FIGURE 3. Another closed path in $\mathcal{P}$ (left), and its deformation.

4.2. Not simply connected. If you have any loop in the $\mathbb{R}^2$, given by a continuous map $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(0) = \gamma(1)$, then you can deform it continuously, while keeping $\gamma(0)$ and $\gamma(1)$ fixed, to a single point—for instance by setting $\gamma_t(s) = \gamma(0) + (1-t)(\gamma(s) - \gamma(0))$. At time $t = 0$ this is the original loop, and for $t = 1$ it is the loop mapping all of $[0,1]$ to $\gamma(0)$, and for all t we have $\gamma_t(0) = \gamma_t(1) = \gamma(0) = \gamma(1)$.

By contrast, when you try to deform the loop in Figure 2 to a point, you will have a hard time. On the other hand, the loop in Figure 3 *can* be continuously deformed to a point. Where is the difference? Well, one can prove that given any loop $\gamma : [0,1] \to \mathbb{P}^2(\mathbb{R})$ and given a $p_0 \in \pi^{-1}(\gamma(0))$ (for which there are two choices), there is a unique continuous path $\gamma' : [0,1] \to S^2$ such that $\gamma'(0) = p_0$ and $\pi(\gamma'(t)) = \gamma(t)$ for all t. However, this γ' need not "close up", that is, it need not be a loop: $\gamma'(1)$ is either equal to p_0 or to $-p_0$ (as it maps to $\gamma(0)$). If it is equal to p_0, then one can deform γ' to a point in S^2—though this requires some work! The image will then deform to a point in $\mathbb{P}^2(\mathbb{R})$. But if $\gamma'(1) = -p_0$ then, deforming γ continuously while keeping $\gamma(0), \gamma(1)$ fixed and lifting the deformation to γ'_t with $\gamma'_t(0) = p_0$, we will have $\gamma'_t(1) = -p_0$ for all t. Indeed, by continuity the endpoint cannot suddenly jump from $-p_0$ to p_0. This proves that γ in Figure 2 cannot be deformed into a point.

The upshot of this is first, that $\mathbb{P}^2(\mathbb{R})$ behaves topologically very different from $\mathbb{R}^2$, and second, that there is a well-defined map from loops in $\mathbb{P}^2(\mathbb{R})$ to $\{-1, 1\}$, sending a loop to 1 if its lift to S^2 closes up, and to -1 otherwise. Of course none of this is very rigorous, and one has to do quite some work to make it so. But here is the main idea that you should remember from it.

> **Philosophy.** Associating to a topological object (here $\mathcal{P}$) an algebraic object (here $\mathbb{Z}/2\mathbb{Z}$), which captures qualitative information about the space, is extremely useful in mathematics. For instance, the idea has far-reaching generalisations in knot theory.

4.3. Embedding graphs in the projective plane. By now you must have learned *Kuratowski's theorem*, which states that a(n undirected) graph is non-planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$. Here *planar* means that you *embed* it into the Euclidean plane, i.e., draw it in the Euclidean plane without intersecting edges. (More formally, it means that there are no injective continous maps from the graph—with an appropriate topology—to the Euclidean plane.)

But you *can* embed K_5 and $K_{3,3}$ in $\mathcal{P}$! Indeed, you can even restrict the edges to be projective line segments, i.e., segments of elements from $\mathcal{L}$. For $K_{3,3}$ this is shown in Figure 4.

EXERCISE 4.1. Show that you can also embed K_5 into $\mathcal{P}$. What about K_6?