Applied algebraic geometry: tensor decomposition

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Iarrobino-Kanev’s algorithm

Input: an $f \in S^d V$ sufficiently small. Output: `fail` or a decomposition of f as a linear combination of pure powers.

- Compute $Q := \ker C_f^e \subseteq S^e V^*$ with $e = \lceil \frac{d}{2} \rceil$.
- Compute the joint zero set $Z \subseteq \mathbb{P}V$ of the elements of Q, regarded as degree-e polynomials on V.
- If Z is a finite set of reduced points $[v_1], \ldots, [v_k]$, then try to solve the linear system $f = \sum_{i=1}^k c_i v_i^d$ for $c_1, \ldots, c_k \in K$. If successful, return the decomposition, otherwise return `fail`.
- Otherwise, return `fail`.

Theorem (special case): If d is odd and $k \leq \binom{n-1+d-e}{d-e}$ and f is a general element of $\overline{kX}$, then the decomposition of f into k pure powers is unique, and the above algorithm finds it.

Proof of Iarrobino-Kanev's theorem

• Write $f = v_1^d + \cdots + v_k^d$ with $(v_1, \ldots, v_k) \in V^k$ general.

• Then for $g \in S^e V^*$, $C_f^e g = d(d-1)\cdots(d-e+1) \sum_i g(v_i) v_i^{d-e}$.

• By the bound on k and generality of the v_i, the v_i^{d-e} are linearly independent in $S^{d-e}V$.

• Hence $C_f g = 0$ if and only if $g(v_i) = 0$ for all i; in other words, Q is the degree-e piece of the vanishing ideal of $[v_1], \ldots, [v_k]$.

• Now apply the following lemma to $s = d - e = e - 1$ (using that d is odd). □

Lemma: Let $S = \{[v_1], \ldots, [v_k]\}$ be general in $\mathbb{P}^{n-1}$, and s such that $\binom{n-1+s}{s} \geq k$. Then the vanishing ideal of S is generated in degree $\leq s+1$, and each piece of degree $\geq s+1$ has the same vanishing set S.

Proof of Lemma

- First for $S \subseteq K^{n-1}$:
- $\pi : K[x_1, \ldots, x_{n-1}] \to K[S] \cong \overbrace{K \times K \times \cdots \times K}^{k}.$
- $M = \{\text{monomials of total degree} \leq s\}$, so $|M| = \binom{n-1+s}{s} \geq k$
- S general $\rightsquigarrow \pi(M)$ spans $K[S]$, so for every monomial u of degree $s+1$ there is a $g_u \in \langle M \rangle$ such that $u - g_u \in \ker \pi$.
- So $\ker \pi$, vanishing ideal of S, generated in degree at most $s+1$.

- Projectively (may assume that S is disjoint from hyperplane where $x_n = 0$): suppose that $f \in K[x_1, \ldots, x_n]$ is homogeneous and vanishes on $S \subseteq K^{n-1} \subseteq \mathbb{P}^{n-1}$

- Write $f = x_n^a g$ where g still vanishes on S and $x_n \nmid g$; $r := \deg g$.
- Set $f' := g(x_1, \ldots, x_{n-1}, 1) \in K[x_1, \ldots, x_{n-1}]$, also of degree r.

- By above, can write $f' = \sum_i a_i f_i$ where $a_i, f_i \in K[x_1, \ldots, x_{n-1}]$, $f_i \in \ker \pi$ of degree $r_i \leq s+1$, and $\deg(a_i) + \deg(f_i) \leq \deg f' =: r$.

- Then $g = x_n^r f'(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}) =$

$$\sum_i \underbrace{(x_n^{r-r_i} a_i(\tfrac{x_1}{x_n}, \ldots, \tfrac{x_{n-1}}{x_n}))}_{\text{hom polynomial}} \underbrace{(x_n^{r_i} f_i(\tfrac{x_1}{x_n}, \ldots, \tfrac{x_{n-1}}{x_n}))}_{\substack{\text{hom polynomial vanishing on } S \\ \text{of degree } r_i \leq s+1.}}$$

$\square$

There are many more results in symmetric tensor decomposition, due to Mourrain, Brachat, Tsigaridas, Ottaviani, Oeding, Comon, Lim, …

Open problem (Comon)

Let $t \in V^{\otimes d}$ be symmetric. Is its rank as an ordinary tensor equal to the symmetric rank of its image in S^d as discussed above?

Orthogonal tensor decomposition

Singular value decomposition
Every symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be written as $A = \sum_{i=1}^{k} \pm v_i v_i^T$ where $v_1, \ldots, v_k$ pairwise orthogonal in $\mathbb{R}^n$.

Consider $d \geq 3$ and V an n-dimensional real vector space with a positive-definite inner product. Can every tensor in S^dV be written as $\sum_{i=1}^{k} \pm v_i^d$ where $v_1, \ldots, v_k \in V$ pairwise orthogonal?

In general not: dimension of set of elements on the right is $\binom{n+1}{2}$, on the left it is $\binom{n-1+d}{d}$. This is the same for $d = 2$, but typically not for $d > 2$.

Theorem (Boralevi-D-Horobeţ-Robeva)
The locus of *orthogonally decomposable tensors* in S^dV is a real-algebraic variety defined by quadratic polynomials.

Odeco tensors

There are also variants for ordinary tensors (where quadratic equations still suffice) and alternating tensors (where cubic equations are needed), and over $\mathbb{C}$ (where cubic and in the alternating case also quartic equations are needed).

Proof for $d = 3$

- A tensor $f \in S^3V$ has the catalecticant map $C_f^2 : S^2V^* \to V$.
- the inner product $(.|.)$ yields an identification $V \to V^*, v \mapsto (.|v)$.
- so f defines a bilinear multiplication $\mu_f : V \times V \to V$.

Concretely, if $f = v^3$, then the multiplication is
$u \cdot w := (u|v)(w|v)v$, and this is extended linearly to general
$f \in S^3V$. Note commutativity, and also $(uw|x) = (ux|w)$. (*)

- now suppose that $f = \sum_i v_i^3$, where v_i are pairwise orthogonal.

- Then compute $(u \cdot w) \cdot v = (\sum_i (u|v_i)(w|v_i)v_i) \cdot v = \sum_i \sum_j (u|v_i)(w|v_i)\underbrace{(v_i|v_j)}_{= 0 \text{ unless } i = j}(v|v_j)v_j = \sum_i (u|v_i)(w|v_i)(v|v_i)\|v_i\|^2 v_i = u \cdot (w \cdot v)$.

- Conversely, assume that the multiplication associated to f is associative.

- By $(*)$ the orthogonal complement of any ideal in the algebra V is itself an ideal. So can decompose V as orthogonal direct sum of simple ideals V_i. Accordingly, $f \in \bigoplus S^3 V_i$, so suffices to prove that $\dim V_i = 1$ for all i. W.l.o.g. V is already simple.

- suppose multiplication is not 0, let $x \in V$ be such that $L_x : y \mapsto xy$ is nonzero. Show $\ker L_x$ is an ideal in V: if $xy = 0$ then $x(yz) = (xy)z = 0$. So L_x is invertible by simplicity.

- Now define a new multiplication $*$ on V by $u * v := L_x^{-1}(uv)$. This is commutative, and also associative (check!), and has x as unit element.

- Summarising, V is a commutative, associative, $\mathbb{R}$-algebra with 1, which moreover is simple. Only possibilities: $V \cong \mathbb{R}$ or $V \cong \mathbb{C}$. But the latter has no compatible inner product $(.|.)$. So $\dim V = 1$.

- So f is odeco iff the multiplication associated to f is associative. This is a list of quadratic equations: must have $\mu_f(\mu_f(u, v), w) = \mu_f(u, \mu_f(v, w))$ for all $u, v, w \in V$. □

Open problem

Do these quadratic equations generate the ideal of the set of odeco tensors?