Applied algebraic geometry: tensor decomposition

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Equations for higher rank

Theorem (Qi, 2014) $\overline{3X}$ is the zero set of degree-4 equations. (4×4-minors of flattenings do not suffice, see $3 \times 3 \times 3$-tensors)

Theorem (D-Kuttler)
For every infinite K and every k the set $\overline{kX} \subseteq V_1 \otimes \cdots \otimes V_d$ is defined by polynomials of degree at most some $N(k)$, which does not depend on d or the dimensions of the V_i.

Known values (in char zero??)

k	0	1	2	3	4	k
$N(k)$	1	2	3	4	≥ 9	$\geq k+1$

Open problems

- (easy?) Is the minimal $N(k)$ increasing with k?
- (hard?) Is tensor rank multiplicative: $\mathrm{rk}(t_1 \otimes t_2) = \mathrm{rk}(t_1) \cdot \mathrm{rk}(t_2)$?

Remarks on symmetry

$G := \prod_i \mathrm{GL}(V_i)$ acts on $V_1 \otimes \cdots \otimes V_d$ by
$(g_1, \ldots, g_d)v_1 \otimes \cdots \otimes v_d = (g_1 v_1) \otimes \cdots \otimes (g_d v_d)$

Set $n_i := \dim V_i$. Then $\dim G = \sum_i n_i^2$, while $\dim \bigotimes_i V_i = \prod_i n_i$. So for dimension reasons, G must have infinitely many orbits on tensors if $d \geq 3$ and the n_i moderately large.

Set $X = \{\text{pure tensors}\}$, of dimension $1 - d + \sum_i n_i$.

It turns out that G has finitely many orbits on $\overline{3X}$ (Buczynski-Landsberg), and this is used by Qi. But (I think) there are already infinitely many orbits on $\overline{4X}$.

My theorem with Kuttler shows that finitely many equations suffice up to the action of G, plus permuting factors, plus flattenings.

We work over algebraically closed K of characteristic zero.

Definition (d-th symmetric power)
V a finite-dimensional vector space, then S^dV is the quotient of $V^{\otimes d}$ by the subspace spanned by all $v_1 \otimes \cdots \otimes v_d - v_{\pi(1)} \otimes \cdots \otimes v_{\pi(d)}$ for $v_1, \ldots, v_d \in V$ and $\pi \in S_d$.

Can write $v_1 \cdots v_d$ for image of $v_1 \otimes \cdots \otimes v_d$ (order doesn't matter); S^dV is canonically isomorphic to the space of homogeneous polynomials of degree d on V^*, via $v_1 \cdots v_d \mapsto (x \mapsto \prod_i x(v_i))$.

Lemma Given any basis $v_1, \ldots, v_n$ of V, the set
$\{v^\alpha := v_1^{\alpha_1} \cdots v_n^{\alpha_n} \mid d_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^n \alpha_i = n\}$ is a basis of S^dV.

Remark Dual notion: the subspace of $V^{\otimes d}$ consisting of all tensors stable under S_d, i.e., symmetric tensors. The map $V^{\otimes d} \to S^d$ restricts to an isomorphism on this subspace (but here we use characteristic zero; for symmetrising you need to divide by $d!$)

For $f = v_1 \cdots v_d \in S^d V$ and $e \leq d$ we have a linear map $C_f^e : S^e V^* \to S^{d-e} V$ determined by $C_f(x_1 \cdots x_e) = \sum_{\varphi:[e]\to[d] \text{ injective}} (\prod_{i=1}^e x_i(v_{\pi(i)})) \cdot \prod_{j \notin \text{im}\pi} v_j$. Extends to general f.

If $f = v^d$, then $\text{im} C_f^e = \langle v^{d-e} \rangle$, so C_f^e has rank ≤ 1 for each e.

Conversely, consider $U = \text{im} C_f^{d-1} \subseteq V$. Choose a basis $v_1, \ldots, v_k$ of U and extend with $v_{k+1}, \cdots, v_n$ of V. Write $f = \sum_\alpha c_\alpha v_\alpha$. Suppose that $c_\alpha \neq 0$ and let i be such that $\alpha_i > 0$. Applying C_f to $x^{\alpha - e_i}$ where $x = (x_1, \ldots, x_n)$ is the dual basis yields (some prod of binomials times) $c_\alpha v_i$, so $v_i \in U$ and $i \leq k$. Conclusion: $f \in S^d U$. In particular, if $\dim U \leq 1$ then $f = v^d$ for some $v \in V$.

Conclusion

$X := \{f \mid \exists v \in V : f = v^d\} \subseteq S^d V$ is a Zariski-closed cone defined by the vanishing of the 2×2-minors of C_f^{e-1}. □

Symmetric rank

Set $X := \{v^d \mid v \in V\} \subseteq S^d V =: T$.

Exercise
Show that X spans T.

So we can speak of the X-rank/border rank of an element of $S^d V$.
This is also called the *symmetric rank* of a symmetric tensor.

In this case, most secant varieties are non-defective:

Theorem (Alexander-Hirschowitz) $\overline{kX}$ has dimension $\min\{kn, \binom{n-1+d}{d}\}$ except in the following cases:

- $d = 2, 2 \leq k \leq n-1$
- $d = 3, n = 5, k = 7$
- $d = 4, (n, k) \in \{(3, 5), (4, 9), (5, 14)\}$

When $\dim kX = kn$, a general $f \in \overline{kX}$ has a finite number of decompositions as $v_1^d + \cdots + v_k^d$.

When in addition $k(n+1) < \dim T$, there is typically just 1 decomposition up to permuting the terms (the two exceptions are rank 9 in S^6K^3 and rank 8 in S^4K^4, where # decompositions is 2).
If $k(n+1) = \dim T$ this uniqueness sometimes holds, but is expected usually *not* to hold.

How to *find* such a decomposition, say when we know it is unique? The following method, due to Iarrobino-Kanev, works for sufficiently small k. There are improvements due to Oeding-Ottaviani and others.

Iarrobino-Kanev's algorithm

Input: an $f \in S^d V$ sufficiently small. Output: `fail` or a decomposition of f as a linear combination of pure powers.

• Compute $Q := \ker C_f^e \subseteq S^e V^*$ with $e = \lceil \frac{d}{2} \rceil$.

• Compute the joint zero set $Z \subseteq \mathbb{P}V$ of the elements of Q, regarded as degree-e polynomials on V.

• If Z is a finite set of reduced points $[v_1], \ldots, [v_k]$, then try to solve the linear system $f = \sum_{i=1}^k c_i v_i^d$ for $c_1, \ldots, c_k \in K$. If successful, return the decomposition, otherwise return `fail`.

• Otherwise, return `fail`.

Theorem (special case): If d is odd and $k \leq \binom{n-1+d-e}{d-e}$ and f is a general element of $\overline{kX}$, then the decomposition of f into k pure powers is unique, and the above algorithm finds it.