Applied algebraic geometry: tensor decomposition

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Recall: semi-algebraic sets

Definition
Take $K = \mathbb{R}$. A *semi-algebraic set* in the $\mathbb{R}$-vector space T is a finite union of sets of the form $\{t \mid f_1(t) = \ldots = f_m(t) = 0, h_1(t), \ldots, h_n(t) > 0\}$ where the f_i and the h_j are polynomials.

Theorem (Tarski-Seidenberg)
The image of a semi-algebraic set under a polynomial map is again semi-algebraic.

In particular, if $X \subseteq T$ Zariski-closed cone, then kX is semi-algebraic!

Lemma
If $S \subseteq T$ is a semi-algebraic set, then, in the *Euclidean* topology, we have $\overline{S}^0 \subseteq \overline{S^0}$

Generic rank versus typical ranks

Now start with T over $\mathbb{R}$, and $X \subseteq T$ Zariski-closed cone.

Definition
$r \in \mathbb{Z}_{\geq 0}$ is a *typical* X-rank if the locus of rank$-r$ elements in T contains an open ball in the Euclidean topology.

Example
For $T = (\mathbb{R}^2)^{\otimes 3}$ and $X = \{\text{pure tensors}\}$ have seen that $2, 3$ are typical ranks.

Theorem (Bernardi-Blekhermann- Ottaviani, 2015)
The typical X-ranks form an interval $\{r_0, r_0 + 1, \ldots, r_1\}$ where r_0 is the generic X-rank.

Typical ranks, continued

Proof

• The generic rank r_0 is typical: for $x_1, \dots, x_{r_0} \in X$ (smooth and) sufficiently general the derivative of the addition map is onto T, so $r_0 X$ contains an open ball around $x_1 + \dots + x_{r_0}$.

• If r is a typical rank, then $rX \setminus (r-1)X$ contains a Euclidean-open ball, and hence its Zariski closure is T, so $r \geq r_0$.

• Now suppose that $r+1$ is *not* a typical rank. Then, with Euclidean topological operations, $((r+1)X)^0 \subseteq \overline{rX}$ and hence $((r+1)X)^0 \subseteq \overline{rX}^0$, and since rX is semi-algebraic this is contained in $\overline{(rX)^0} =: Y$.

• Now we argue that Y is stable under adding elements of X to it. Indeed, for each $x \in X$ we have $x + (rX)^0 \subseteq ((r+1)X)^0$ (because the first set is open) and this is contained in Y. But then also $x + Y \subseteq Y$.

• Hence $Y = T$ and no integer $> r$ is a typical rank. $\square$

Take K algebraically closed, characteristic zero, and $X \subseteq T$ closed cone spanning T.

Lemma

If the ideal of X has no nonzero poly of degree $< d_0$, then the ideal of $\overline{kX}$ has no nonzero poly of degree $< d_0 + (k-1)$.

Proof

By induction on k. For the induction step, let $f \in K[T]$ be nonzero, of degree $d > 0$, and vanish on $(k+1)X$. Let $v \in X$ be such that the directional derivative g of f is not identically zero on T (use that X spans T). Then g is a nonzero poly of degree $d-1$.

Now expand $f(y + tv) = f(y) + tg(y) + \cdots$. The left-hand side vanishes identically for $(y, t) \in \overline{kX} \times K$, hence g vanishes on $\overline{kX}$. $\square$

$T = V_1 \otimes \cdots \otimes V_d$ and $X = \{$ pure tensors $\}$, then X is the zero set of the 2×2-subdeterminants of flattenings. So $\overline{2X}$ has no equations of degree < 3. But in fact, 3 suffices:

Theorem (Landsberg-Manivel, 04)
$t \in T$ is in $\overline{2X}$ if and only if all $\flat_{I,J} t$ has rank ≤ 2 for all (I, J).

Proof of $\Leftarrow$ for nonzero t
• Replace V_i by the image of $\flat_{i,[d]-i} t : (\bigotimes_{j \neq i} V_j)^* \to V_i$, and remove 1-dimensional ones $\rightsquigarrow$ w.l.o.g. each V_i has dimension 2.
• For $d = 3$ have already seen the result. More precisely, if t has vanishing Cayley hyperdet, then check that
$t \in V_1 \otimes v_2 \otimes v_3 + v_1 \otimes V_2 \otimes v_3 + v_1 \otimes v_2 \otimes V_3$
for some $v_i \in V_i$. (It lies on the *tangential variety* to X.)
• Indeed, they prove this last expression for elements of $\overline{2X} \setminus 2X$ for general d.

• By d=3 case $\rightsquigarrow$ two cases:
Case I can write $t = u \otimes v \otimes s + u' \otimes v' \otimes s'$ with each pair $u, u' \in V_1$, $v, v' \in V_2$ lin ind, and $s, s' \in \bigotimes_{i>2} V_i$ nonzero
• assume s not pure; then $\flat_{I,J} s = A \otimes B + A' \otimes B'$ for some $I \cup J = \{3, \ldots, d\}$ and lin ind pairs A, A', B, B'.
• must have $s' \in \langle A, A' \rangle \otimes \langle B, B' \rangle$. Then
$t = u \otimes v \otimes A \otimes B + u \otimes v \otimes A' \otimes B' + u' \otimes v' \otimes s'$
and then $v \otimes A, v \otimes A', v' \otimes$ something $\in \operatorname{im} \flat_{(1J),(2I)} t$, contradiction, so s, s' pure, so $t \in 2X$.

Case 2 can write $t = u' \otimes v \otimes s + u \otimes v' \otimes s + u \otimes v \otimes s'$.
In this case, prove that s is pure: $s = v_3 \otimes \cdots \otimes v_d$ and that $s' \in \sum_{i>2} v_1 \otimes \cdots \otimes V_i \otimes \cdots \otimes v_p$, so t lies on the tangential variety of X.
$\square$