Applied algebraic geometry: tensor decomposition

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Some dimension counting

We'll mostly assume K algebraically closed today.

Example 1: pure tensors

$X = \{v_1 \otimes \cdots \otimes v_d \mid v_i \in V_i\} \subseteq V_1 \otimes \cdots \otimes V_d$ has dimension $1 + \sum_i(-1 + \dim V_i)$: indeed, have a surjective map
$\varphi : V_1 \times \cdots \times V_d \to X, (v_1, \ldots, v_d) \mapsto v_1 \otimes \cdots \otimes v_d$,
so X is irreducible and $\dim X = \sum_i \dim V_i - \dim \varphi^{-1}(x)$ for $x \in X$ general (in some open dense subset of X). Now $\varphi^{-1}(v_1 \otimes \cdots \otimes v_d) = \{(t_1 v_1, \ldots, t_d v_d) \mid t_1 \cdots t_d = 1\}$ if all $v_i \neq 0$.

Example 2: bounded-rank matrices

$Y = \{\text{matrices of rank} \leq k\} \subseteq K^{m\times n}$ is closed, irreducible since it is the image of matrix multiplication $K^{m\times k} \times K^{k\times n} \to K^{m\times n}$.
Assuming $k \leq m, n$, $\operatorname{codim}_{K^{m\times n}} Y = (n-k)(m-k)$.

Assume K alg closed. Dominant map $X^k \to \overline{kX}$ so $\dim(\overline{kX}) \leq \min\{\dim T, k \dim X\}$, the *expected dimension*. A sufficiently general element of T is expected to have rank $\lceil \dim T / \dim X \rceil$. For many triples (X, T, k) these statements are true with equality. But $\exists$ counterexamples, even in the tensor setting, and hard to classify.

Example (Strassen, Ottaviani): Let $X \subseteq (K^3)^{\otimes 3} = T$ be the pure tensors, so $\dim X = 3 + 3 + 3 - 2 = 7$ in a 27-dimensional space. Expect $\overline{4X} = T$, but in fact it is a hypersurface of degree 9:

$$\varphi : (K^3)^{\otimes 3} \to K^3 \otimes (K^3 \otimes K^3) \otimes K^3) \cong (K^3 \otimes K^3) \otimes (K^3 \otimes K^3)$$

$$u \otimes (x, y, z) \otimes v \mapsto u \otimes \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix} \otimes v$$

φ linear, sends pure tensors to 9×9-matrices of rank 2 $\rightsquigarrow$ sends tensors of rank ≤ 4 to matrices of rank ≤ 8, so det vanishes on $\overline{4X}$!

Recall: tangent spaces

Assume K alg closed. If $X \subseteq K^n$, irreducible, has vanishing ideal generated by $f_1, \ldots, f_r$, then form the Jacobi-matrix $J := (\frac{\partial f_i}{\partial x_j})_{i,j}$. The rank of J at a general point $x_0 \in X$ is constant, and equal to $n - \dim X$. The kernel of $J(x_0)$ is the *tangent space* $T_{x_0}X$. It has dimension at least $\dim X$, and if equality holds then X is called *smooth* at x_0. Smooth points form an open, dense subset of X.

A morphism $\varphi : X \to Y$ of algebraic varieties has a *derivative* $d_{x_0}\varphi : T_{x_0}X \to T_{y_0}Y$ at $x_0 \in X$. If $X \subseteq K^n$ and $Y \subseteq K^m$ and φ is the restriction of some polynomial map $K^n \to K^m$, then this is given by the linear part of that polynomial map near x_0.

Example

$X = \{\text{rank-}\leq 1 \text{ tensors}\} \rightsquigarrow$

$T_{v_1 \otimes \cdots \otimes v_d} X = \sum_{i=1}^d v_1 \otimes \cdots \otimes V_i \otimes \cdots \otimes v_d$.

Terracini's lemma

Suppose char$K = 0$, K alg closed, X irred. For $(x_1, \ldots, x_k) \in X^k$ sufficiently general set $y := x_1 + \ldots + x_k$. Then $T_y\overline{kX} = \sum_i T_{x_i}X$.

Proof

Since char$K = 0$, the derivative of the dominant addition map $X^k \to \overline{kX}$ is surjective for $(x_1, \ldots, x_k)$ in an open dense subset. $\square$

Exercise

In this setting, set $d_k := \dim \overline{kX}$. Prove $d_{k+1} - d_k \leq d_k - d_{k-1}$.

Very degenerate example

$T = K^m \otimes K^n$, $X = \{\text{rank} \leq 1 \text{ matrices}\}$ of dimension $m + n - 1$, $kX = \overline{kX} = \{\text{rank} \leq k \text{ matrices}\}$ of dimension $k^2 + (m-k)k + (n-k)k$

Recall: constructible sets

Definition
T vsp over K, algebraically closed, $U \subseteq T$ is *locally closed* if it is an intersection of a closed set and an open set. It is *constructible* if it is a finite union of locally closed subsets.

Chevalley's theorem ("quantifier elimination")
$\varphi : T \to T'$ polynomial map, $U \subseteq T$ constructible $\Rightarrow \varphi(U)$ constructible. (Also true in varieties more general than T.)

In particular, each kX is constructible, and contains an open dense subset of $\overline{kX}$. The minimal k such that $\overline{kX} = T$ is the *generic X-rank*.

Theorem
$K = \mathbb{C} \Rightarrow$ Zar. closure of a constr. set equals its Euclidean closure.

K alg closed, $X \subseteq T$ closed cone spanning the vector space T. The maximal X-rank of an element of T is often > the X-rank of a sufficiently general element of T. How much larger?

Theorem (Blekherman-Teitler, 2014)
Maximal rank $\leq 2\cdot$ generic rank.

Proof. There is $U \subseteq T$ open, dense subset such that all $u \in U$ have the generic rank. Let $t \in T$. Then $t - U$ is open, dense, hence intersects U, so $t - u_1 = u_2$ for some $u_1, u_2 \in U$. □

Example
$\mathrm{char} K = 0$, $T = \{$homogeneous polynomials in x, y of degree $d\}$, $\dim T = d + 1$, $X = \{(ax + by)^d \mid a, b \in K\}, \dim X = 2$
generic rank is $\lceil (d + 1)/2 \rceil$, maximal rank is d, attained by $x^{d-1}y$.