Applied algebraic geometry: tensor decomposition

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Observation

K a field. Then $A \in K^{m\times n}$ has $\mathrm{rk}A \le k \Leftrightarrow A$ can be written as $\sum_{i=1}^{k} u_i v_i^T$ with $u_i \in K^m, v_i \in K^n$.

$\Leftarrow$ each $u_i v_i^T$ has rank 1 and $\mathrm{rk}(B + C) \le \mathrm{rk}(B) + \mathrm{rk}(C)$
$\Rightarrow$ induction on k: if $A \neq 0$, take nonzero $u \in \mathrm{colspace}(A)$, w such that $Aw = u$, and v such that $\ker A \subseteq \ker v^T$ and $v^T w \neq 0$. Then $A' := A - \frac{uv^T}{v^T w}$ has $\ker A' \supseteq Kw + \ker A$ so $\mathrm{rk}A' < \mathrm{rk}A$. □

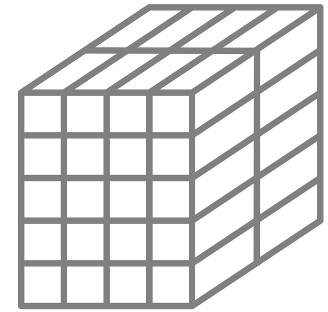
Many variations

- matrices with structure (symmetric, skew)
- want the u_i pairwise $\perp$, and also the v_i (SVD over $\mathbb{R}$ or $\mathbb{C}$)
- *approximations* by low-rank matrices (SVD)

Central question: *How does this all generalise to tensors?*

What's a tensor?

Answer 1: a multidimensional array of numbers.
$(a_{ijk})_{i\in[5],j\in[4],k\in[2]}$



Answer 2: an element of $T := V_1 \otimes \cdots \otimes V_d$ for $V_1, \ldots, V_d$ f.d. vector spaces over K

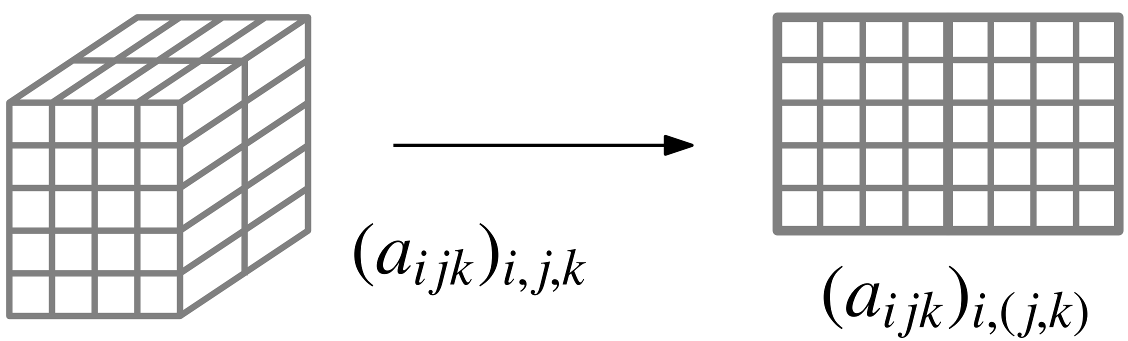
Recall

Elements of T are formal linear combinations of symbols $v_1 \otimes \cdots \otimes v_d$ modulo the space spanned by elements of the form $v_1 \otimes \cdots \otimes (v_i + u_i) \otimes \cdots \otimes v_d - v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_d - v_1 \otimes \cdots \otimes u_i \otimes \cdots \otimes v_d$ and $v_1 \otimes \cdots \otimes (cv_i) \otimes \cdots \otimes v_d - c(v_1 \otimes \cdots \otimes v_d)$

- $\{\text{linear } f : T \to W\} \cong \{\text{multilin } f : V_1 \times \cdots \times V_d \to W\}$
- B_j basis of $V_j \Rightarrow B_1 \otimes \cdots \otimes B_d$ basis of T *(relates answers 1,2)*
- $V_1^* \otimes \cdots \otimes V_d^* \cong T^*, (x_1 \otimes \cdots \otimes x_d)(v_1 \otimes \cdots \otimes v_d) = \prod_i x_i(v_i)$

Flattening and pure tensors

Given a partition $[d] = I \cup J$, have a natural map $\flat_{I,J} : T \to T_I \otimes T_J$ with $T_I := \bigotimes_{i\in I} V_i$, given by $v_1 \otimes \cdots \otimes v_p \mapsto (\otimes_{i\in I} v_i) \otimes (\otimes_{i\in J} v_j)$. Here we forget the tensor product structure on T_I, T_J. Similarly with more factors.



In general, $U^* \otimes V \cong \mathrm{Hom}_K(U, V)$, $x \otimes v \mapsto (u \mapsto x(u)v)$.

Take $t \in T$. For each $i \in [d]$, let U_i :=image of t as linear map $T^*_{[d]-i} \to V_i$. Then $t \in \bigotimes_i U_i$ and the U_i are minimal with this property. All $\dim U_i = 1 \Leftrightarrow t = u_1 \otimes \cdots \otimes u_d$, some nonzero u_i.

Definition t is called *pure* if $t = u_1 \otimes \cdots \otimes u_d$ for some u_i.

Proposition
Pure tensors in T form a Zariski-closed subset X defined by quadratic polynomials.

Proof
For $d = 2$ these are the rank ≤ 1 matrices, defined by 2×2-subdeterminants. For $d > 1$, t pure iff $\flat_{[d]-i,i} t$ pure for all i. □

Remark $|K| = \infty \rightsquigarrow 2 \times 2$-dets of flattenings generate ideal(X).

Definition
T any vector space, $X \subseteq V$ Zariski-closed cone spanning T. Then
- $kX := \{x_1 + \cdots + x_k \mid x_i \in X\}$;
- $\mathrm{rk}_X t := \min\{k \mid v \in kX\}$ the *X-rank* of v;
- $\overline{kX}$ is the *k-th secant variety of* X; and
- $\mathrm{brk}_X t := \min\{k \mid v \in \overline{kX}\}$ the *X-border rank* of v.

For $X \subseteq T$ above, this is *tensor (border) rank*.

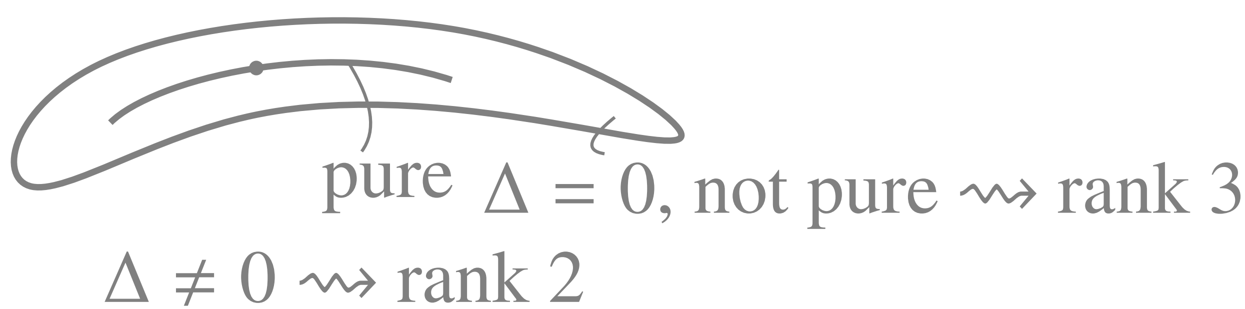
Case study: $2 \times 2 \times 2$-tensors

$t \in K^2 \otimes K^2 \otimes K^2$, write $t = e_1 \otimes A + e_2 \otimes B$ with A, B 2×2-matrices

- suppose $\mathrm{rk} \flat_{1,23} t = 2$, i.e., A, B linearly independent
- then $\mathrm{rk} t = 2$ iff $\exists C, D$ of rank 1 and u, v such that $t = u \otimes C + v \otimes D$
- iff $\langle A, B \rangle$ contains two linearly independent rank-1 matrices
- iff the discriminant $\Delta(t)$ of the quadratic polynomial $\det(xA + yB)$ is a nonzero square—this is *Cayley's hyperdeterminant*:

$$\Delta(t) = a_{2,2}^2 b_{1,1}^2 - 2a_{2,1}a_{2,2}b_{1,1}b_{1,2} + a_{2,1}^2 b_{1,2}^2 - 2a_{1,2}a_{2,2}b_{1,1}b_{2,1} - 2a_{1,2}a_{2,1}b1, 2b_{2,1} + 4a_{1,1}a_{2,2}b_{1,2}b_{2,1} + a_{1,2}^2 b_{2,1}^2 + 4a_{1,2}a_{2,1}b_{1,1}b_{2,2} - 2a_{1,1}a_{2,2}b_{1,1}b_{2,2} - 2a_{1,1}a_{2,1}b_{1,2}b_{2,2} - 2a_{1,1}a_{1,2}b_{2,1}b_{2,2} + a_{1,1}^2 b_{2,2}^2$$

Picture for K alg closed:



For $K = \mathbb{R}$ have
$\Delta > 0 \rightsquigarrow$ rank 2,
$\Delta < 0 \rightsquigarrow$ rank 3.