APPLIED ALGEBRAIC GEOMETRY, SHEET 8

For $n \in \mathbb{Z}_{\geq 0}$ let $R_n := K[x_{ij} \mid i \in [k], j \in [n]]$. Every (strictly) map $\pi : [m] \to [n]$ yields a K-algebra homomorphism $\pi : R_m \to R_n$ determined by $\pi x_{ij} = x_{i\pi(j)}$. In the lecture we proved a finiteness results for sequences of ideals $I_n \subseteq R_n$ such that $\pi I_m \subseteq I_n$ for all increasing maps π. We want to prove a variant for certain subrings of R_n.

For each $n \in \mathbb{Z}_{\geq 0}$ let P_n be a subring of R_n such that:

(1) $\pi P_m \subseteq P_n$ for each increasing $\pi : [m] \to [n]$, and
(2) for each n, R_n decomposes as $P_n \oplus Q_n$, where Q_n is a P_n-submodule of R_n and where each increasing $\pi : [m] \to [n]$ maps Q_m into Q_n.

Exercises:

(1) Prove that, under these assumptions, if for each n, $J_n \subseteq P_n$ is an ideal such that all increasing $\pi : [m] \to [n]$ maps J_m into J_n, then there exists an n_0 such that for all $n \geq n_0$ the ideal J_n is generated by the images πJ_m where $m \leq n_0$ and $\pi : [m] \to [n]$ is strictly increasing. (Deduce this from the corresponding statement for R.)

(2) Let P_n be the subring of R_n generated by all monomials $x_{1j_1} x_{2j_1} \cdots x_{kj_k}$ for $j_1, \ldots, j_k \in [n]$. Show that P_n-submodules Q_n as above exist.

(This last statement was the missing ingredient in the proof of the independent set theorem: P_n is the coordinate ring of the variety of rank-one tensors of format $n \times \cdots \times n$ (k factors).)

Handed out on November 11, to be handed in on November 17.