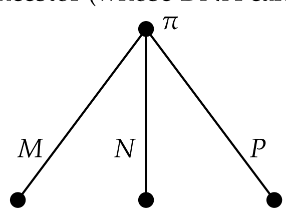
APPLIED ALGEBRAIC GEOMETRY, SHEET 3

(1) Let K be algebraically closed of characteristic zero, $X \subseteq K^n$ a closed, affine cone spanning K^n, and suppose that $d_0 \geq 2$ is the lowest degree of homogeneous polynomials vanishing on X. Show that for each $k \geq 1$, a polynomial $f \in K[x_1, \ldots, x_n]$ of degree $d_0 + (k-1)$ vanishes on $\overline{kX}$ if and only if all of its $(k-1)$-st order derivatives vanish on X.

(2) Let $t \in V_1 \otimes V_2 \otimes V_3 \otimes V_4$ and assume that $\mathrm{rk}\, \flat_{12,34} t$ and $\mathrm{rk}\, \flat_{13,24} t$ are both equal to 2. Find a (good) upper bound on the rank of t.

(3) Here is a simple model for evolution along the following tree evolutionary tree of three extant species with a common (extinct) ancestor (whose DNA cannot be measured):



For each position in the (aligned) DNA-string of the ancestor, a tetrahedral die is flipped with side labels A, C, G, T and respective probabilities $\pi = (\pi_A, \pi_C, \pi_G, \pi_T) \in \mathbb{R}^4$. This ancestral nucleotide then mutates into a nucleotide for each of the extant species, according to transition probabilities given by the stochastic matrices M, N, P (nonnegative entries, row sums 1). These mutations are independent. The outcome is a triple in $\{A, C, G, T\}^3$.

(a) Verify that the probability t_{ijk} of outcome (i, j, k) equals

$$\sum_{l \in \{A,C,G,T\}} \pi_l M_{l,i} N_{l,j} P_{l,k}$$

(b) Determine the maximal rank that the tensor $(t_{ijk})_{i,j,k}$ can have. (Hint: you also need to argue that the upper bound you find is attained; for this try a parameter count.)

(This simple model is a building block for larger *phylogenetic tree models*, and polynomial equations defining the Zariski closure of the model are/were the so-called *Salmon problem*.)

Handed out on October 6, to be handed in on October 13.