APPLIED ALGEBRAIC GEOMETRY, SHEET 2

(1) Let K be algebraically closed, and $X \subseteq K^{m\times n}$ the closed subvariety of matrices of rank at most k, which is assumed $\leq m, n$.
   (a) Prove that X is irreducible.
   (b) Prove that the matrices of rank equal to k form a dense, open subset U of X.

   Let GL_n be the group of invertible $n \times n$-matrices. It is also an n^2-dimensional affine variety, and the multiplication and inverse are morphisms of affine varieties, i.e. it is an algebraic group. Let $G = \mathrm{GL}_n \times \mathrm{GL}_m$ act on X via $(g, h)A = gAh^{-1}$.
   (c) Show that U equals the G-orbit of the matrix
   $$J := \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$
   in which I_k is the $k \times k$-identity matrix.
   (d) Determine the stabiliser of J in G, and compute its dimension.
   (e) Derive from this the dimension of X.

(2) Let K be algebraically closed of characteristic zero, $X \subseteq K^n$ a Zariski-closed cone, and for $k \in \mathbb{Z}_{\geq 0}$ set $d_k := \dim \overline{kX}$. Prove that $d_{k+1} - d_k \leq d_k - d_{k-1}$ for all $k \geq 1$, i.e., the sequence $(d_k)_k$ is concave. (Hint: use Terracini's lemma.)

(3) Let K be algebraically closed of characteristic zero, $T = \{a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d \mid a_0, \ldots, a_d \in K\}$ the space of *binary forms* of degree d, and $X = \{(ax + by)^d \mid a, b \in K\}$ the variety of d-th powers. We want to show that the maximal X-rank of an element of T equals d.
   (a) Let $\sigma_i(a_1, \ldots, a_d) := \sum_{I\subseteq[d],|I|=d} \prod_{i\in I} a_i$ be the i-th *elementary symmetric function* in $a_1, \ldots, a_d$. Prove that the map $K^d \to K^d, a \mapsto (\sigma_1(a), \ldots, \sigma_d(a))$ is surjective. (Hint: polynomial factorisation.)
   (b) Let $\pi_i(a_1, \ldots, a_d) := a_1^i + \cdots + a_d^i$ be the i-th *power sum*. Prove that the map $K^d \to K^d, a \mapsto (\pi_1(a), \ldots, \pi_d(a))$ is also surjective. (Hint: you may use that the π_i are polynomials in the σ_j and vice versa—look up *Newton's identities*, e.g. on Wikipedia.)
   (c) Conclude that over K every univariate polynomial of the form $dx^d + b_1x^{d-1} + \ldots + b_d$ is a sum of d polynomials of the form $(x + c)^d$, and that the maximal X-rank of an element of T is at most d.
   (d) * Show that $x^{d-1}y$ does not have rank $< d$. (The * means I have to re-think about it myself ....)

Handed out on September 30, to be handed in on October 6.