APPLIED ALGEBRAIC GEOMETRY, SHEET 10

(1) Verify that for a jointly Gaussian vector $X = (X_1, \dots, X_n)$ defined via a DAG G, indeed X_i is conditionally independent of $X_{\mathrm{nd}(i) \setminus \mathrm{pa}(i)}$ given $X_{\mathrm{pa}(i)}$.

(2) Prove the following strengthening of the Sullivant-Talaska result: Given a directed Gaussian graphical model based on a DAG G on $[n]$, and given $A, B \subseteq [n]$ (possibly of different cardinalities), and given a nonnegative integer r, then the following two statements are equivalent:

   (a) The submatrix $\Sigma_{A,B}$ has rank $\leq r$ for all choices of the parameters; and

   (b) there exist subsets $C_A, C_B \subseteq [n]$ such that $|C_A| + |C_B| \leq r$ and such that every trek from A to B either passes C_A on its way up or C_B on its way down (where the top of the trek counts both as belonging to the up part and as belonging to the down part).

   (For this latter part, look up Menger's theorem.)

Handed out on November 25, to be handed in on December 1.