APPLIED ALGEBRAIC GEOMETRY, SHEET 1

(1) Let K be a field and let $A \in K^{n\times n}$ be skew-symmetric. Show that the matrix rank $\operatorname{rk} A$ satisfies

$$\operatorname{rk} A = 2 \min \left\{ k \mid \exists u_1, \ldots, u_k, v_1 \ldots, v_k \in K^n : A = \sum_i (u_i v_i^T - v_i u_i^T) \right\}.$$

(2) Assume that K is algebraically closed and let $A \in K^{n\times n}$ be symmetric. Show that

$$\operatorname{rk} A = \min \left\{ k \mid \exists v_1, \ldots, v_k : A = \sum_i v_i v_i^T \right\},$$

and discuss to what extent this fails for $K = \mathbb{R}$.

(3) Determine the rank and border rank of the tensor

$$t = e_1 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_2 + e_2 \otimes e_1 \otimes e_2 - e_2 \otimes e_2 \otimes e_1 \in K^2 \otimes K^2 \otimes K^2$$

in the cases where

(a) $K = \mathbb{R}$;
(b) K is algebraically closed of characteristic $\neq 2$;
(c) K is algebraically closed of characteristic 2; and
(d) K is finite of odd characteristic.

Handed out on September 23, to be handed in on September 29.