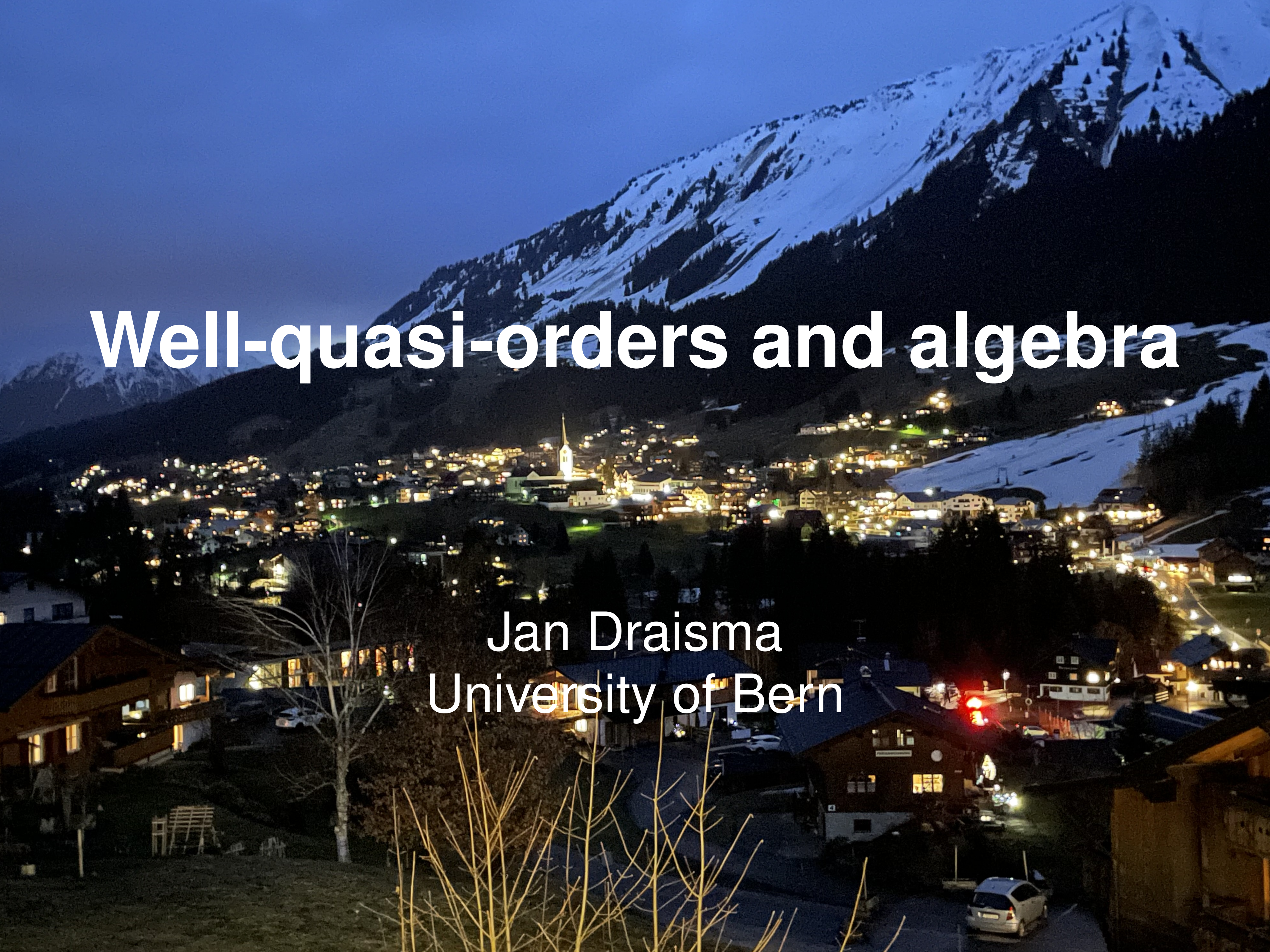

Well-quasi-orders and algebra

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Definition. A *well-quasi-order* on a set A is a transitive relation $\preceq$ s.t. for all $a_1, a_2, ...$ there exist $i < j$ with $a_i \preceq a_j$.

(Non-)examples

- Any transitive relation on a *finite* set A is a wqo.
- The standard $\leq$ on $\mathbb{N} = \{1, 2, ...\}$ is a wqo; $|$ is not.
- The standard $\leq$ on $\mathbb{Z}$ is not a wqo: $-1, -2, ...$
- A is infinite $\Rightarrow$ $=$ is not a wqo: infinite antichains.

Lemma. If $\preceq$ is a wqo on A and $a_1, a_2, ... \in A$, then $\exists i_1 < i_2 < ...$ with $a_{i_1} \preceq a_{i_2} \preceq ...$

Dickson's Lemma. $\preceq_i$ a wqo on A_i for $i = 1, 2 \rightsquigarrow$ $(a, b) \preceq (c, d) :\Leftrightarrow (a \preceq_1 c$ and $b \preceq_2 d)$ is a wqo on $A_1 \times A_2$.

Higman's lemma

$A^* = \bigcup_{d=0}^{\infty} A^d$ is the set of words on A

Lemma (Higman)
If $\preceq$ is a wqo on A, then $\preceq$ on A^* defined by $(a_1, \ldots, a_d) \preceq (b_1, \ldots, b_e) :\Leftrightarrow \exists$ strictly increasing $\pi : [d] \to [e]$ with $\forall i : a_i \preceq b_{\pi(i)}$ is a wqo.

Proof

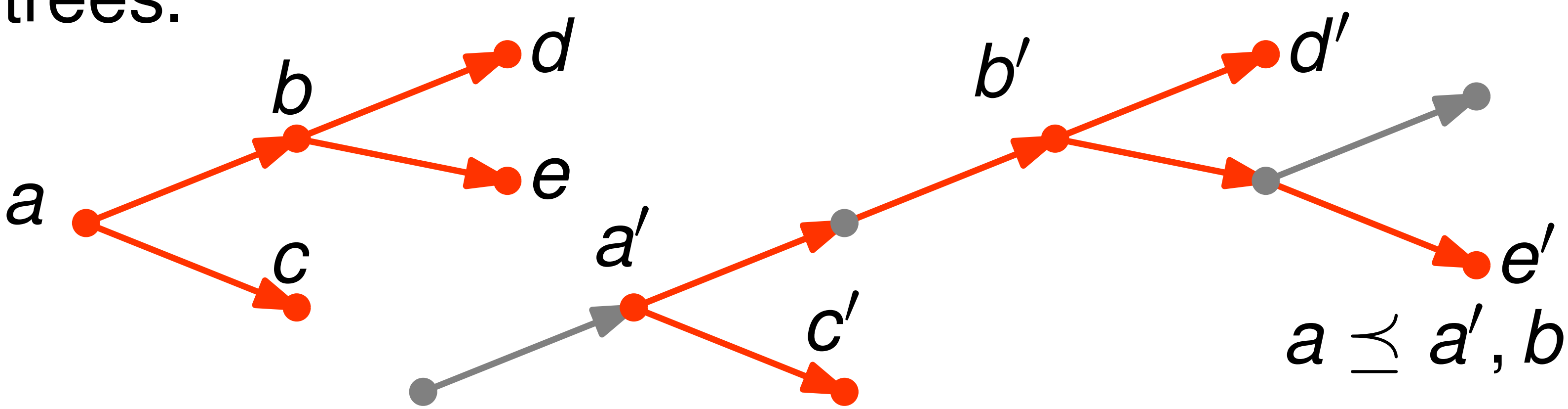
• If not, take a *bad sequence*, i.e., $w_1, w_2, \ldots \in A^*$ with $\nexists i < j : w_i \preceq w_j$; *minimal* in the following sense: w_k is shortest among all bad sequences starting with $w_1, \ldots, w_{k-1}$.

• Write $w_i = (a_i, u_i)$ and find $i_1 < i_2 < \ldots$ with $a_{i_1} \preceq a_{i_2} \preceq \ldots$

• Now $w_1, \ldots, w_{i_1-1}, u_{i_1}, u_{i_2}, \ldots$ is a bad sequence $\rightsquigarrow$ contradiction. $\square$

Theorem (Kruskal)

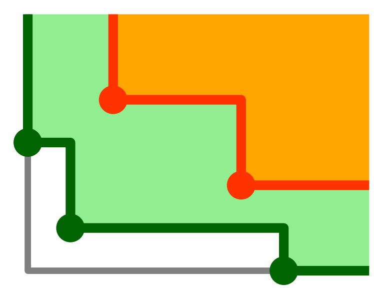
Generalisation of Higman's lemma to rooted, *A*-labelled trees.



$$a \preceq a', b \preceq b', \ldots$$

Theorem (Maclagan)

The set of monomial ideals in $K[x_1, \ldots, x_n]$ is well-quasi-ordered by $I \preceq J := I \supseteq J$.



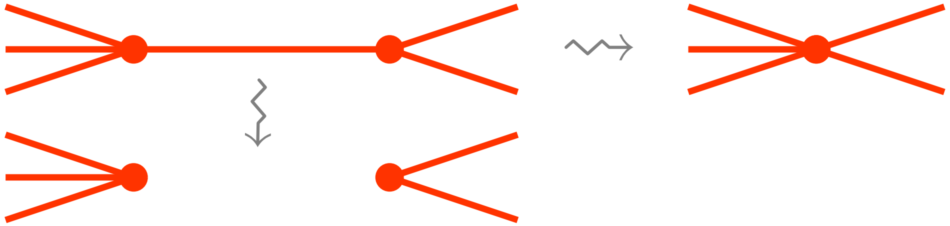
A new well-quasi-order

Tensor restriction fheorem (Blatter-D-Rupniewski)
$\mathbb{F}_q$ a finite field, $d \in \mathbb{Z}_{\geq 0}$ $V_1, V_2, \ldots$ f.d. $\mathbb{F}_q$-spaces, $T_i \in V_i^{\otimes d}$. Then $\exists i < j, \varphi \in \mathrm{Hom}(V_j, V_i) : \varphi^{\otimes d} T_j = T_i$.

Corollary
Every property of order-d tensors over $\mathbb{F}_q$ preserved under applying linear maps can be tested in polynomial time.

Multilinear analogue of:

Theorem (Robertson-Seymour)
The minor order on finite, undirected graphs is a wqo.



Prototypical use of a wqo in algebra

K a field (or Noetherian ring), fixed throughout

Hilbert's basis theorem
Every ideal I in $K[x_1, \dots, x_n]$ is finitely generated.

Proof template:
• monomials in $x_1, \dots, x_n$ are wqo wrt $x^\alpha | x^\beta$ (Dickson);
• hence with respect to any monomial order $\leq$ the set $\mathrm{lm}(I) = \{\mathrm{lm}(f) \mid f \in I \setminus \{0\}\}$ has finitely many $|$-minimal elements: $\mathrm{lm}(f_1), \dots, \mathrm{lm}(f_k)$.
• $f_1, \dots, f_k$ generate I (division with remainder). $\square$

Linear algebra over categories

Representations of a category

Definition. C a category $\rightsquigarrow$ a *C-module over K* is a (covariant) functor $M : C \to \mathrm{Mod}_K$, i.e.:

- $\forall S : M(S)$ is an K-module;
- $\forall \pi \in \mathrm{Hom}_C(S, T)$: $M(f) : M(S) \to M(T)$ is K-linear;
- and $M(1_S) = 1_{M(S)}$ and $M(\sigma \circ \pi) = M(\sigma) \circ M(\pi)$.

Remarks

- C-modules over K form an abelian category.
- Many natural notions, such as *finitely generated*.
- Each $M(S)$ is a representation of $\mathrm{Aut}_C(S)$.

Example

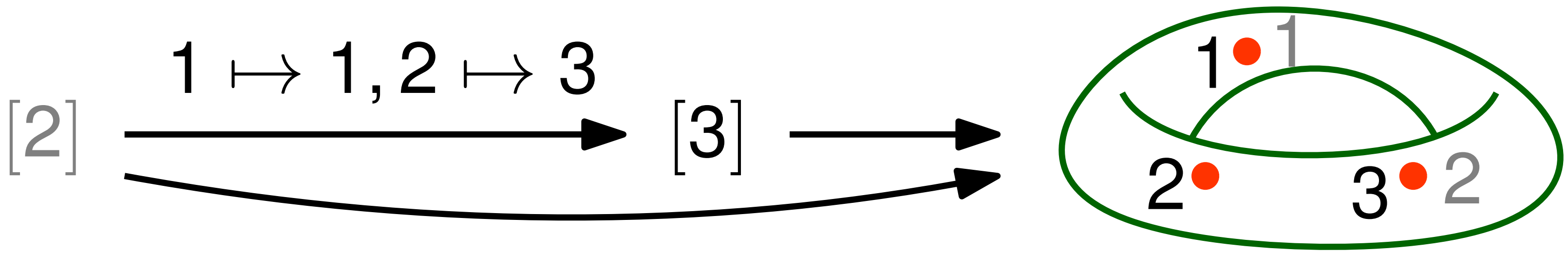
$C = \mathbf{FI}$: Finite sets with Injections

$M(S) = K \cdot \binom{S}{2}$, generated by $1 \cdot \{1, 2\} \in M([2])$

FI-modules in the wild

Example

X a manifold $\rightsquigarrow$ Conf_X : $\mathbf{FI}^{\mathrm{op}} \to$ manifolds, defined by $\mathrm{Conf}_X(S) = \{\text{injective maps } S \to X\}$; the *pure configuration space* of X.



Theorem (Church-Ellenberg-Farb)

Fix $p \geq 0$. Under mild conditions on X, $H^p(\mathrm{Conf}_X, \mathbb{Q})$ is a finitely generated **FI**-module over $\mathbb{Q}$.

Nice consequences, e.g. $\dim_{\mathbb{Q}}(H^p(\mathrm{Conf}_X([n]), \mathbb{Q}))$ is polynomial in n for $n \gg 0$, and splits into a fixed number of S_n-representations (*representation stability*).

Finitely generated **FI**-modules

Theorem (Church-Ellenberg-Farb). Every sub-**FI**-module V of a finitely generated **FI**-module M is finitely generated.

Same proof template (Sam-Snowden)

• Work with **OI**: sets $[d], d \in \mathbb{Z}_{\geq 0}$ with increasing maps.

• M is a quotient of $P = P_{d_1} \oplus \cdots \oplus P_{d_k}$, where $P_d(S) = K \cdot \mathrm{Hom}_{OI}([d], S)$; suffices to prove for $M = P$.

• For basis elements $\pi \in \mathrm{Hom}_{\mathbf{OI}}([d], S) \subseteq P_d(S)$ and $\sigma \in \mathrm{Hom}_{\mathbf{OI}}([d], T)$, write $\pi \preceq \sigma$ if $\exists \varphi \in \mathrm{Hom}_{\mathbf{OI}}(S, T) : \sigma = \varphi \circ \pi$.

• This is a wqo on the basis in each P_d (Higman's lemma for $A = \{0, 1\}$ with $=$), hence on the basis in P.

• Choose an **OI**-compatible linear order on the basis in each $P(S)$. Then $\exists$ finitely many $v_i \in V(S_i)$ s.t. $\forall S$, $\forall v \in V(S) \exists i : \mathrm{Im}(v_i) \preceq \mathrm{Im}(v)$. These generate $V \subseteq P$. $\square$

Similar finiteness results

Sam-Snowden call **OI** a *Gröbner category*, and **FI** a *quasi-Gröbner category*.

FS: Finite sets and Surjective maps
FS$^{\text{op}}$: the opposite category

Theorem (Sam-Snowden)
FS$^{\text{op}}$ is quasi-Gröbner. Hence any sub-**FS**$^{\text{op}}$-module of a finitely generated **FS**$^{\text{op}}$-module is finitely generated.

Proof: (not so easy) exercise: find an ordered version of **FS**$^{\text{op}}$ and a suitable wqo, etc.

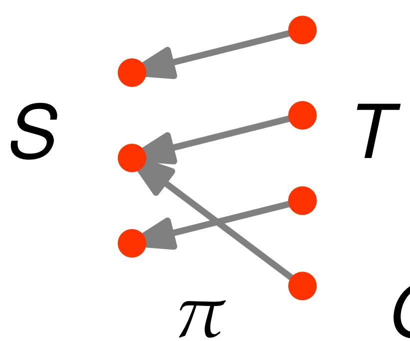
FI-modules appear naturally throughout math.
FS$^{\text{op}}$-modules not so much, but . . .

Let $\mathbb{F}_q$ be a finite field, $\mathbf{Vec}_{\mathbb{F}_q}$ the category of finite-dimensional vector spaces over $\mathbb{F}_q$.

Corollary (Putman, Sam, Snowden)
Finitely generated $\mathbf{Vec}_{\mathbb{F}_q}$-modules M over K are Noetherian.

Proof

- Have a functor $Q : \mathbf{FS}^{\text{op}} \to \mathbf{Vec}_{\mathbb{F}_q}$, $Q(S) = \mathbb{F}_q S$.



$$\rightsquigarrow \quad Q(\pi) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- Show that $M \circ Q$ is a finitely generated $\mathbf{FS}^{\text{op}}$-module: every $T \times$bounded matrix over $\mathbb{F}_q$ is of the form $Q(\pi) \cdot A$ for some bounded $\times$ bounded matrix A. □

Commutative algebra over categories

Definition. A *C-algebra over K* is a functor from C to (commutative, unital) K-algebras.

Natural notions of ideals and Noetherianity.

Coordinate rings of matrix spaces as FI-algebras

- $B(S) := K[x_{ij} \mid i, j \in S]$
- $A_c(S) := K[x_{ij} \mid i \in [c],\ j \in S]$

Examples

- The ideals $I_k \subseteq A$ and $J_k \subseteq B$ generated by all $k \times k$-determinants is finitely generated in both B (by $k+1$ elements) and in A (by $\binom{c}{k}$ elements).
- B is not Noetherian.

Theorem (Daniel Cohen, 1987)
The **FI**-algebra $A_c : S \mapsto K[x_{ij} \mid i \in [c], j \in S]$ is Noetherian.

Many, many applications and follow-up work:
- the independent set theorem (Hillar-Sullivant)
- biv. Hilbert series (Nagel-Römer, Krone-Leykin-Snowden)
- co-dimension, projective dimension, regularity (Van Le-Nagel-Nguyen-Römer)
- moment varieties of mixtures of products (Alexandr-Kileel-Sturmfels, ...)

Theorem (D-Eggermont-Farooq-Meier)
For any homomorphism $R \to A_c$ of finitely generated **FI**-algebras, the image closure of $\mathrm{Spec}(A_c) \to \mathrm{Spec}(R)$ is set-theoretically defined by finitely many equations.

Rank-one tensors as an **FS**-variety

Fix n.

For any finite set S, define $V(S) := (K^n)^{\otimes S}$, and for any surjective $\pi : T \to S$ define $V(\pi) : V(T) \to V(S)$ by $\bigotimes_{j \in T} v_j \mapsto \bigotimes_{i \in S} \bigodot_{j \in \pi^{-1}(i)} v_j$, where $\odot =$.

The locus $X(S) \subseteq V(S)$ of rank-1 tensors is an **FS**-variety.

Theorem (D-Oosterhof)
Coordinate ring $S \mapsto K[X(S)]$ is a Noetherian **FS**$^{\text{op}}$-algebra. (Proof uses Maclagan's theorem.)

$\rightsquigarrow$ Ideals of *iterated toric fibre products* of undirected discrete graphical models stabilise as the number of factors tend to ∞ (builds on work by Rauh-Sullivant and Kahle-Rauh).

The tensor restriction theorem

Theorem (Blatter-D-Rupniewski)
Let $P : \mathbf{Vec}_{\mathbb{F}_q} \to \mathbf{Vec}_{\mathbb{F}_q}$ be *any* functor of finite length. Then the $\mathbf{Vec}_{\mathbb{F}_q}^{\mathrm{op}}$-algebra $V \mapsto \mathbb{F}_q^{P(V)} = SP(V)^* / \langle f^q - f \mid f \rangle$ is Noetherian.

Corollary
Given $p_i \in P(V_i), i = 1, 2, ...,$ there exist $i < j$ and $\varphi : V_j \to V_i$ with $P(\varphi)p_j = p_i$. (Notation: $p_i \preceq p_j$)

Proof
Let I_i be the ideal of functions that vanish on all p with $p_j \not\preceq p$ for all $j = 1, \ldots, i$. Then $I_{j-1} = I_j$ for some j and hence $p_i \preceq p_j$ for some $i < j$. $\square$

Thank you!