Tropical Brill-Noether theory

Jan Draisma

Tropical Day at Warwick, 17 February 2011

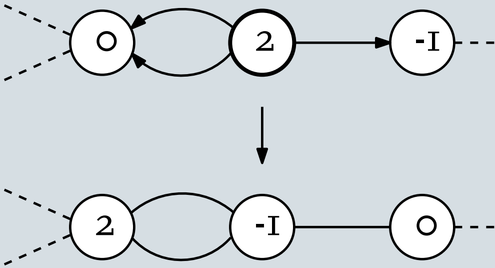
(j.w.w. Cools-Robeva-Payne and van der Pol)

The B(aker)-N(orin) game on graphs

Requirements
finite, undirected graph Γ
$d \geq 0$ chips
natural number r

Rules
B puts d chips on Γ
N demands $r_v \geq 0$ chips at v with $\sum_v r_v = r$
B wins iff he can *fire* to meet N's demand



Brill-Noether theorems for graphs

$g := e(\Gamma) - v(\Gamma) + 1$ *genus* of Γ
$\rho := g - (r+1)(g-d+r)$

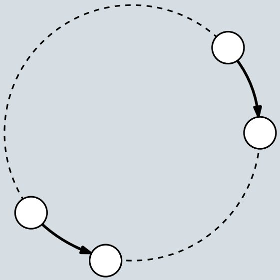
Conjecture (Baker)
1. $\rho \geq 0 \Rightarrow$ B has a winning starting position.
2. $\rho < 0 \Rightarrow$ B may not have one, depending on Γ.
($\forall g \, \exists \Gamma \, \forall d, r : \rho < 0 \Rightarrow$ Brill loses.)

Theorem (Baker)
1. is true if B may put chips at rational points of edges.
(*uses* sophisticated algebraic geometry)

Theorem (Cools-D-Payne-Robeva)
2. is true.
(*implies* sophisticated algebraic geometry)

Chip dragging on graphs

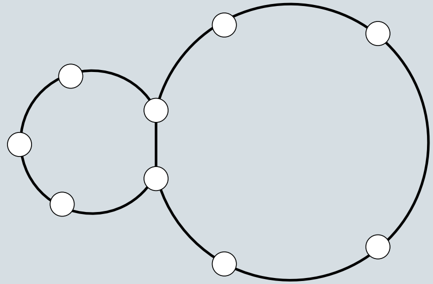
Simultaneously moving all chips along edges, with zero net movement around every cycle.



Lemma
1. Chip dragging is realisable by chip firing.
2. W.l.o.g. B *drags* instead of *firing*.

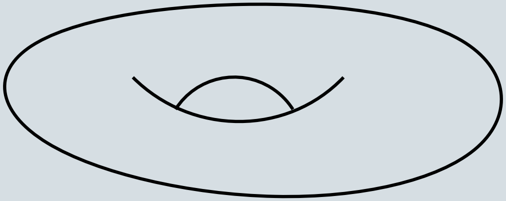
Example 1: Γ a tree
$\rho = g-(r+1)(g-d+r) = -(r+1)(-d+r)$
B wins $\Leftrightarrow \rho \geq 0 \Leftrightarrow d \geq r$

Example 2: a hyperelliptic graph
$d = 2, r = 1$
Who wins?



The B(rill)-N(oether) game on curves

Requirements
compact Riemann surface X
d chips
natural number r



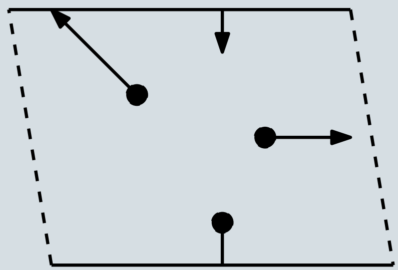
Rules
B puts d chips on X
N demands $r_x \geq 0$ chips at x with $\sum_x r_x = r$
B wins iff he can *drag* to meet N's demand

Chip dragging on curves

Simultaneously moving chips c along paths $\gamma_c : [0,1] \to X$, such that $\sum_c \langle \omega|_{\gamma(t)}, \gamma_c'(t) \rangle = 0$ for all holomorphic 1-forms ω on X.

Lemma
$D = \sum_c [\gamma_c(0)]$ initial position
$E = \sum_c [\gamma_c(1)]$ final position
$\Leftrightarrow E - D$ is *divisor* of meromorphic function on X
drag-equivalent $\sim$ *(linear equivalence)*

Example: torus
only one holomorphic 1-form: $\mathrm{d}z$
condition: $\sum_c \gamma_c'(t) = 0$
when does B win?



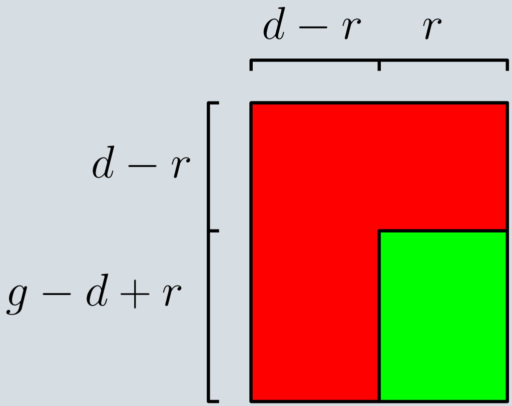
Dimension count

$\omega_1, \dots, \omega_g$ basis of holomorphic 1-forms
$\mathbf{x} = (x_1, \dots, x_d) \in X \times \dots \times X$
$v_i \neq 0$ tangent vector at x_i
$\rightsquigarrow$ matrix $A_{\mathbf{x}} = (\langle \omega_i, v_j \rangle)_{ij} \in \mathbb{C}^{g \times d}$
$(c_1 v_1, \dots, c_d v_d)$ infinitesimal dragging direction $\Rightarrow A(c_1, \dots, c_d)^T = 0$

$\mathbf{x}$ winning for $B \Rightarrow$
dragging $\mathbf{x}$ fills $\geq r$-dimensional variety
where $\ker A$ is $\geq r$-dimensional

conditions on $g \times d$-matrix to have $\geq r$-dimensional kernel: $r(g-d+r)$

for B to have a winning position, "need"
$d - r(g-d+r) \geq r$
$\Leftrightarrow \rho = g - (r+1)(g-d+r) \geq 0$



Brill-Noether theorems for curves

Theorem (Meis 1960, Kempf 1971, Kleiman-Laksov 1972)
$\rho \geq 0 \Rightarrow$ B has a winning position.

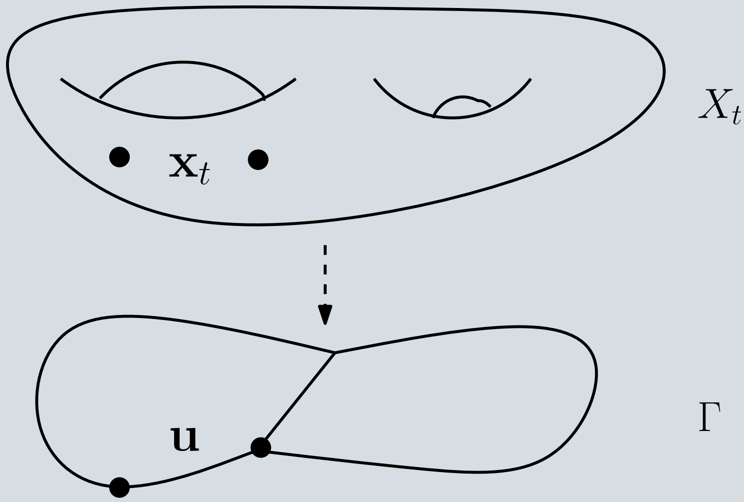
Theorem (Griffiths-Harris 1980)
1. $\rho < 0 \Rightarrow$ B may lose, depending on X.
($\forall g \, \exists X \, \forall d, r : \rho < 0 \Rightarrow$ B loses.)

2. $\rho \geq 0$ and X general
$\Rightarrow \rho = \dim\{$winning directions modulo dragging$\}$

3. $\rho = 0$ and X general
$\Rightarrow \# = \#$ standard tableaux of shape
$(r+1) \times (g-d+r)$ with entries $1, 2, \ldots, g$

Baker's Specialization Lemma

$\{X_t\}_{t \neq 0}$ family of surfaces
$\{\mathbf{x}_t\}_{t \neq 0}$ family of positions
$X_t \rightsquigarrow \Gamma$ for $t \longrightarrow 0$
$\mathbf{x}_t \rightsquigarrow \mathbf{u}$ for $t \longrightarrow 0$



Lemma
$\mathbf{x}_t$ winning on X_t for all $t \neq 0$
$\Rightarrow$ $\mathbf{u}$ winning on Γ.

Consequences

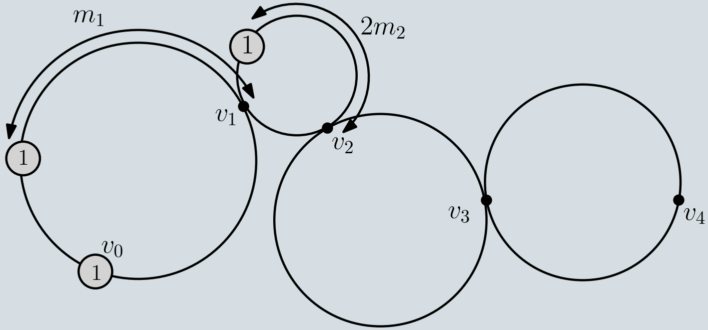
1. Kleiman-Laksov $\Rightarrow$ winning positions on *metric* Γ
2. Cools-D-Payne-Robeva $\Rightarrow$ Griffiths-Harris 1 and 2.

Example of Cools-D-Payne-Robeva

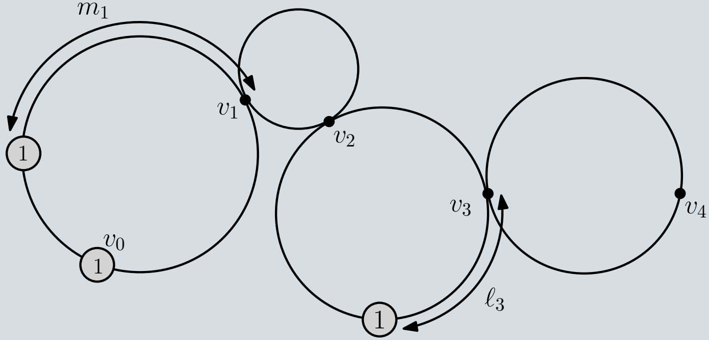
$g = 4, d = 3, r = 1$

$\rightsquigarrow \rho = 0$

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \rightsquigarrow 1, 2, 3, 2, 1$

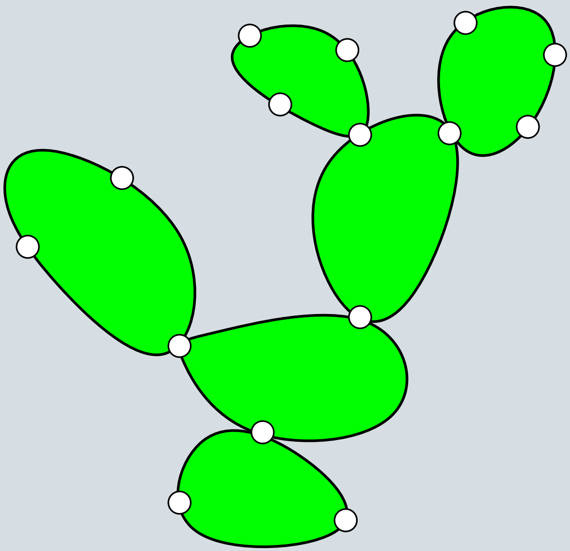


$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \rightsquigarrow 1, 2, 1, 2, 1$



Chips at vertices?

Theorem (van der Pol-D)
$\rho \geq 0$ and Γ a *cactus graph*
$\Rightarrow$ B has winning positions with all chips at vertices.



Future goal:
Understand Kleiman-Laksov for (metric) graphs.