Technische Universiteit
Eindhoven
University of Technology

A tropical approach to secant dimensions

Jan Draisma
j.draisma@tue.nl

October 2011

SIAM conference
Applications of Algebraic Geometry
North Carolina State University

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Slicing the cube

Question
For $n \neq 4$ does $C = \{0,1\}^n$ have a *regular partition* into $C_1, \ldots, C_k, C_{k+1}$ with
- $C_1, \ldots, C_k$ affine bases of $\mathbb{R}^n$ and
- C_{k+1} affinely independent?

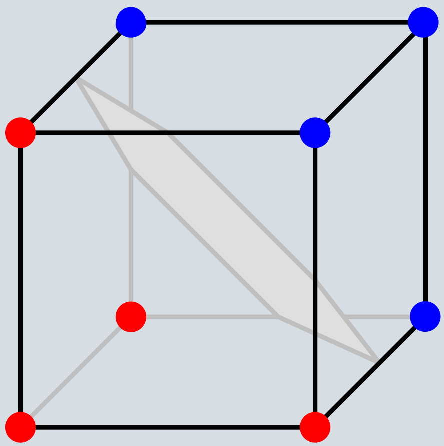
$(k = \lfloor \frac{2^n}{n+1} \rfloor)$

This would imply:

Theorem (Catalisano, Geramita, Gimigliano)
For $n \neq 4$ the n-th Segre power of $\mathbb{P}^1$ has no defective higher secant varieties.

Regular partitions

$f_1, \ldots, f_{k+1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ affine linear
$C_i = \{v \in C \mid f_i(v) < f_{j \neq i}(v)\}$



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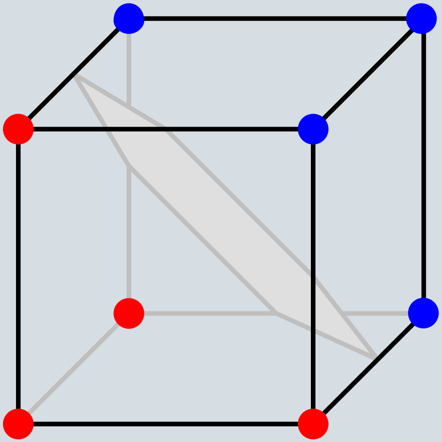

Secant varieties

$\varphi = (x^{\alpha_1}, \ldots, x^{\alpha_p}) : \mathbb{C}^n \to \mathbb{C}^p$
monomial map (can be weakened)
$X := \overline{\operatorname{im} \varphi}$
$kX := \overline{\{u_1 + \ldots + u_k \mid u_i \in X\}}$
k-th *secant variety*
expect $\dim kX = \min\{k \dim X, p\}$
kX called *defective* otherwise

Interesting X
pure tensors (Segre)
powers of linear forms (Veronese)
Segre-Veronese embeddings
pure alternating powers (Plücker)

Theorem (Abo-Ottaviani-Peterson)
Grassmannians of planes are mostly non-defective.

Tropical approach

$C = \{\alpha_1, \ldots, \alpha_p\} \subseteq \mathbb{N}^n$
$f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ linear
$C_i := \{\alpha_q \mid f_i(\alpha_q) < f_{j \neq i}(\alpha_q)\}$

Theorem
$\dim kX$ is at least
$\sum_i \dim \langle C_i \rangle_{\mathbb{R}} =: H(f_1, \ldots, f_k)$.
(The art is to *maximise* H.)

Ciliberto-Dumitrescu-Miranda
degenerations

Develin, D—tropical:
- $\operatorname{Trop} \varphi$ is linear map $\mathbb{R}^n \to \mathbb{R}^p$;
- $\max_{(f_1,\ldots,f_k)} H$ is dimension of *k-th tropical secant variety* of $\operatorname{Trop} X$, *contained* in $\operatorname{Trop} kX$;
- $\dim_{\mathbb{R}} \operatorname{Trop} kX = \dim_{\mathbb{C}} kX$.

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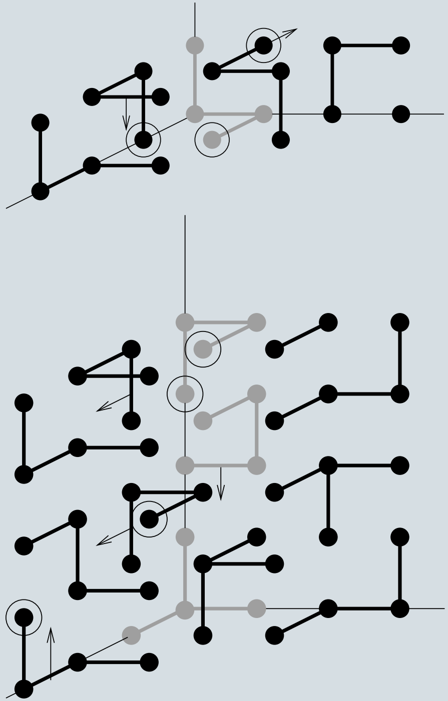
A Segre-Veronese example

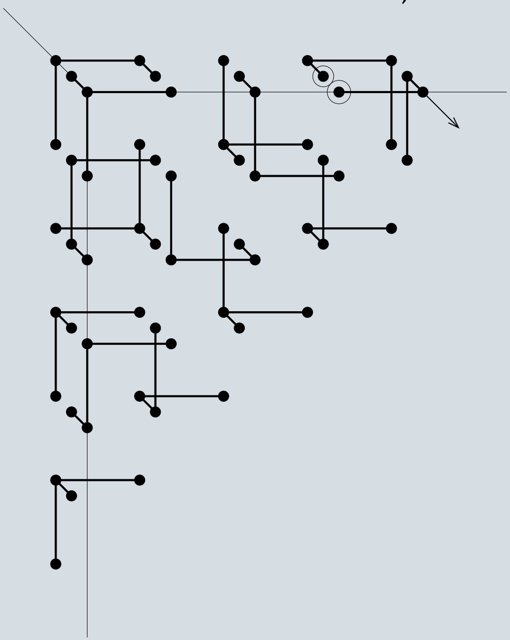
Theorem (Baur-D)
All Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^2$ are non-defective, except an explicit list.



A non-monomial example

F variety of point-line flags in $\mathbb{P}^2$

Theorem (Baur-D)
All SL_3-equivariant embeddings of F into projective spaces are non-defective, except into V_λ with λ once or twice the adjoint weight.



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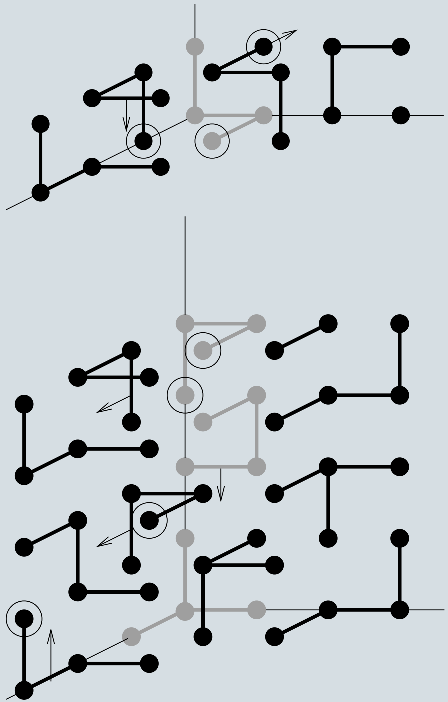

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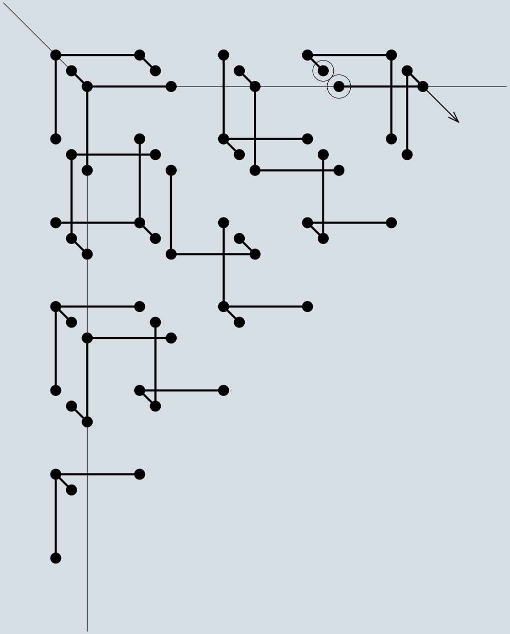

Thank you!

Questions?