Tropical reparameterisations

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SIAM conference
Applications of Algebraic Geometry
North Carolina State University

Technische Universiteit
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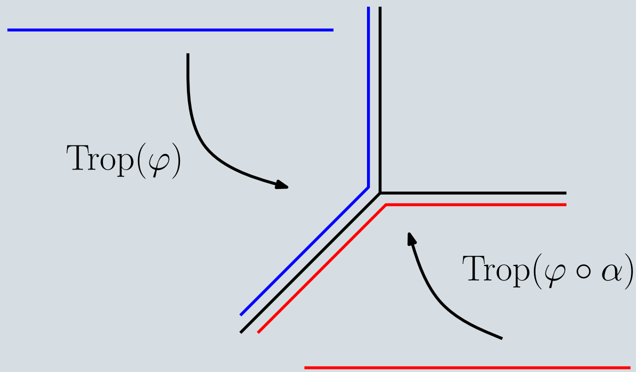
Reparameterising a line

$\varphi : (\mathbb{C}^*) \dashrightarrow (\mathbb{C}^*)^2, t \mapsto (t+1, t)$
$\mathrm{Trop}(\varphi) : \mathbb{R} \to \mathbb{R}^2,$
$t \mapsto (\min\{t, 0\}, t)$
Not surjective!



Coordinate change
$\alpha : \mathbb{C}^* \dashrightarrow \mathbb{C}^*, s \mapsto s - 1$
$\varphi \circ \alpha : s \mapsto (s, s-1)$
$\mathrm{Trop}(\varphi \circ \alpha) : s \mapsto (s, \min\{s, 0\})$

Problem statement

(K, v) non-Archimedean field
$\varphi : (K^*)^m \dashrightarrow (K^*)^n$ rational map
$X := \overline{\operatorname{im}\varphi}$ unirational
$\operatorname{Trop}(\varphi) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$
(replace $c \in K$ by $v(c)$,
$+$ by $\min$, times by addition,
division by subtraction)

Fact
$\operatorname{Trop}(\varphi)$ maps $\mathbb{R}^m$ into $\operatorname{Trop}(X)$.

Question
$\exists\, p \in \mathbb{N}$ and $\alpha : (K^*)^p \dashrightarrow (K^*)^m$ such that $\operatorname{Trop}(\varphi \circ \alpha)$ surjective onto $\operatorname{Trop}(X)$?

If yes, call φ *tropically reparameterisable.*

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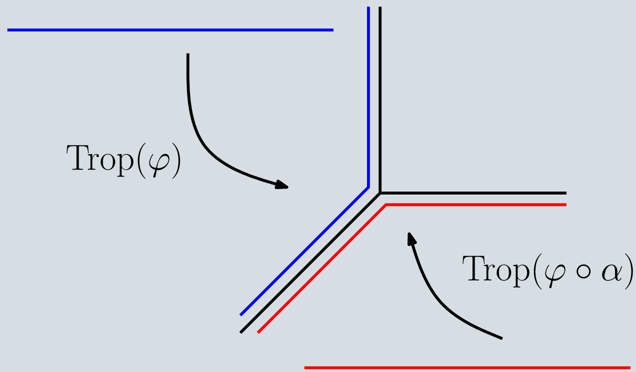

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Known

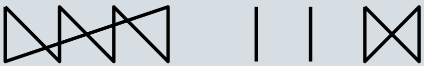
Lemma
- Question is equivalent to: $\exists\, (p_i, \alpha_i),\ i = 1, \ldots, k$ such that $\bigcup_i \operatorname{Trop}(\varphi \circ \alpha_i) = \operatorname{Trop}(X)$?
- W.l.o.g. X is a hypersurface.
- W.l.o.g. φ is birational to X.

(But perhaps not w.l.o.g. both.)
- Need K algebraically closed.

Tropically reparameterisable
- $m = 1$: curves (Speyer)
- φ linear (Yu-Yuster)
- if φ is, then so is $\mu \circ \varphi$ with μ monomial
- $\operatorname{Gr}(2, n)$, rank-2 matrices, …
- Horn uniformisation of A-discriminants a (Dickenstein-Feichtner-Sturmfels)

Singular matrices

$\varphi : (K^*)^{4\times 3} \times (K^*)^{3\times 4} \longrightarrow (K^*)^{4\times 4}$,
$(A, B) \mapsto AB$
$X = V(\det)$
Maximal cones of $\mathrm{Trop}(X)$:

$\binom{4}{0}^2 \frac{4!3!}{2}$ 

$\binom{4}{1}^2 \frac{3!2!}{2}$  $\binom{4}{2}^2 \cdot 2! \cdot \frac{2!1!}{2}$

Observation
For $(i, j) \in [4]^2$ set Y_{ij} := union of cones with full-dimensional projection pr along (i, j)-entry.
$\rightsquigarrow \mathrm{pr} : Y_{ij} \longrightarrow \mathbb{R}^{[4]^2-(i,j)}$ bijective!
$\rightsquigarrow 4^2$ reparameterisations like
$(K^*)^{3\times 3} \times ((K^*)^3)^2 \longrightarrow (K^*)^{4\times 4}$,

$$(A, v, w) \mapsto \begin{bmatrix} A & v \\ w^t & w^t A^{-1} v \end{bmatrix};$$

tropicalisations cover $\mathrm{Trop}(X)$.

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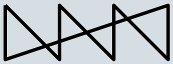
$\binom{4}{0}^2 \frac{4!3!}{2}$  

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Theorem
$\mathrm{Trop}(V(\det))$ is *tropically unirational*: the image of $\mathbb{R}^{n^2 \times (n^2-1)}$ under some tropical rational map.

A local result

$\mathrm{char}\, K = 0$ and $\dim_{\mathbb{Q}} v(K^*)$ finite

Theorem
$y = (y_1, \ldots, y_n) \in \mathrm{Trop}(X)$
very generic on a dimension-d polyhedron P of $\mathrm{Trop}(X)$
Then $\exists \alpha : (K^*)^d \dashrightarrow (K^*)^m$ s.t. $\mathrm{Trop}(\varphi \circ \alpha)$ hits an open neighbourhood of y in P.

Very generic means $\langle y_1, \ldots, y_n \rangle_{\mathbb{Q}}$ mod $v(K^*)$ has $\mathbb{Q}$-dimension d.

Thank you!

Questions?

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