Tropical reparameterisations
(tropically unirational varieties)

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Tropical Geometry
Castro Urdiales

(Based on discussions with Bart Frenk, Filip Cools, Wouter Castryck, Anders Jensen, Bernd Sturmfels, Josephine Yu, . . .)

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Set-up

K algebraically closed field
$v : K^* =: T \to \mathbb{R}$ valuation
(perhaps trivial)
$f \in K(x_1, \ldots, x_m)$
$\rightsquigarrow \mathrm{Trop}(f) : \mathbb{R}^m \to \mathbb{R}$
$\varphi : T^m \dashrightarrow T^n$
$\rightsquigarrow \mathrm{Trop}(\varphi) : \mathbb{R}^m \to \mathbb{R}^n$

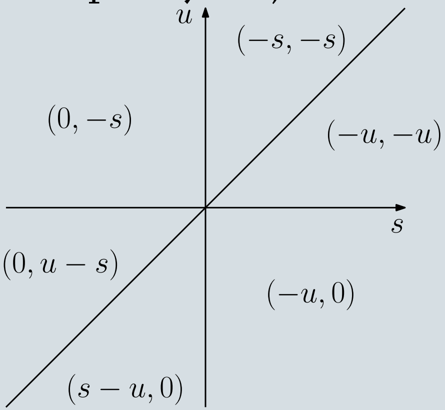
Definition
$X \subseteq T^n$ is *tropically uni-rational* if $\exists p$ $\exists \psi : T^p \dashrightarrow T^n$ s.t.
$\overline{\mathrm{im}\,\psi} = X$ and
$\mathrm{im}\,\mathrm{Trop}(\psi) = \mathrm{Trop}(X)$.

Remarks
Always $\mathrm{im}\,\mathrm{Trop}(\psi) \subseteq \mathrm{Trop}(\overline{\mathrm{im}\,\psi})$.
We allow $p \gg \dim X$.
Call such ψ *tropically surjective*.

Line

$X \subseteq T^2$ defined by $y = x + 1$
$\varphi : T^1 \dashrightarrow T^2,\ t \mapsto (t, t+1)$
not tropically surjective!

But $\psi : T^2 \dashrightarrow X \subseteq T^2$,
$(s, u) \mapsto (\frac{1+s}{u-s}, \frac{1+u}{u-s})$
is tropically surjective.



So X is tropically unirational.
Note: cannot take $p = 1$.

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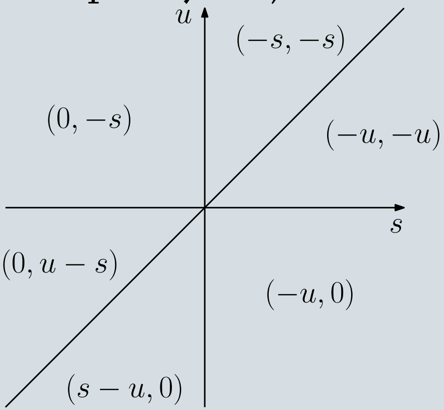

So X is tropically unirational.
Note: cannot take $p = 1$.

Central question

Is every unirational variety tropically unirational?

Lemma
$X \subseteq T^n$ tropically unirational
$\pi : T^n \to T^q$ homomorphism
then $\overline{\pi(X)}$ tropically unirational.
(If ψ is a tropically surjective parameterisation of X, then $\pi \circ \psi$ is tropically surjective to $\overline{\pi(X)}$.)

Theorem (Yu-Yuster)
Linear spaces X are tropically unirational.
(Take matrix ψ with one column for each minimal-support vector in $\overline{X} \subseteq K^n$.)

Some examples…

Example
Affine-linear spaces X are tropically unirational.
(Apply Yu-Yuster to cone $\tilde{X}$ spanned by $X \times \{1\} \subseteq T^{n+1}$, and use homomorphism $\pi : T^{n+1} \to T^n,\ (y,t) \mapsto t^{-1}y$.)

Example (Speyer)
Rational curves X are tropically uni-rational.
(Say X parameterised by $\varphi(t) = (f_1(t), \ldots, f_n(t))$. Factor $f_i(t) = \prod_{j=1}^{l}(t - t_j)^{e_{ij}} \in K(t)$, and note $X = \pi(Y)$ with Y affine-linear parameterised by $(t - t_1, \ldots, t - t_l)$ and π a torus homomorphism.)

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…and some more

Example
Grassmannian $G_{2,n} \subseteq T^{\binom{n}{2}}$ is tropically unirational.
Parameterised by $(u_iu_j(x_i - x_j))_{i<j}$, hence image of linear space under torus homomorphism.

Similarly: A-discriminants (Dickenstein - Feichtner - Sturmfels, Horn Uniformisation), rank-2 matrices, …

Reparameterisations

$X \subseteq T^n$ unirational
$\varphi : T^m \dashrightarrow T^n$ with $\overline{\operatorname{im}\varphi} = X$

Definition
Rational maps $T^p \dashrightarrow T^n$
of the form $\varphi \circ \alpha$ with
$\alpha : T^p \dashrightarrow T^m$
are *reparameterisations* of φ.

Strategy
To prove X tropically unirational, try and find a tropically surjective reparameterisation of φ.

Note: if φ is birational to X, then every dominant $\psi : T^p \dashrightarrow X$ is a reparameterisation of φ.

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Combining reparameterisations

Combination Lemma
Assume $\varphi : K^m \longrightarrow K^n$ regular, $\alpha_i : K^{p_i} \longrightarrow K^m$ regular, $i = 1, 2$
$\alpha_1(0) = \alpha_2(0) = 0$,
and set $\psi_i := \varphi \circ \alpha_i$, $i = 1, 2$.

Then $\operatorname{im}\operatorname{Trop}(\psi_1), \operatorname{im}\operatorname{Trop}(\psi_2) \subseteq \operatorname{im}\operatorname{Trop}(\psi)$, where $\psi := \varphi \circ \alpha$ and $\alpha : K^{p_1+p_2} \longrightarrow K^m$,
$\alpha(u, v) := \alpha_1(u) + \alpha_2(v)$.

($\varphi(\alpha_1(u) + \alpha_2(v)) = \varphi(\alpha_1(u)) + \sum_{j=1}^{p_2} v_j \mu_j(u, v)$.
Given tropical values for u, sufficiently large values for v_j make this tropicalise to $\operatorname{Trop}(\psi_1)(u)$. Similarly for ψ_2.)

From finitely many to one

Proposition
A rational variety $X \subseteq T^n$ is tropically unirational iff $\exists N \exists \psi_i : T^{p_i} \dashrightarrow X, i = 1, \ldots, N$ such that $\bigcup_{i=1}^{N} \operatorname{im} \operatorname{Trop}(\psi_i) = \operatorname{Trop}(X)$.

($\Leftarrow$: Take $\varphi : T^m \dashrightarrow X$ birational, $\alpha_i : T^{p_i} \dashrightarrow T^m$ such that $\psi_i = \varphi \circ \alpha_i$.
Homogenise φ, α_i to homogeneous regular maps $\tilde{\varphi}, \tilde{\alpha}_i$ of positive degree with $\tilde{\varphi} \circ \tilde{\alpha}_i$ parameterising cone $\tilde{X} \subseteq K^{n+1}$. Apply Combination Lemma to obtain a single $\tilde{\psi}$, and dehomogenise to obtain ψ.
Note: ψ automatically dominant.)

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Birational projections

$X \subseteq T^n$ unirational, dimension d
$I \subseteq [n] = \{1, \ldots, n\}, |I| = d$
$\pi_I : T^n \to T^I$ projection

Definition
P d-dimensional polyhedron of $\operatorname{Trop}(X)$ is *I-vertical* if $\dim(\operatorname{Trop}(\pi_I)P) < d$

Proposition
If $\pi_I|_X : X \dashrightarrow T^I$ birational with inverse φ, then $\operatorname{im} \operatorname{Trop} \varphi$ is the union U of all P that are *not* I-vertical.

($\supseteq$: Points u for which $\operatorname{Trop}(\pi_I)(u)$ is not in the corner locus of $\operatorname{Trop} \varphi$ are dense in U and satisfy $\operatorname{Trop}(\varphi)(\operatorname{Trop}(\pi_I)u) = u$.)

Two applications…

Example
$X \subseteq K^n$ d-dimensional affine-linear space.
Then all I, $|I| = d$ with $\dim \pi_I X = d$ (bases of the matroid) satisfy the requirement. Hence X is tropically unirational. (With more work, alternative proof of Yu-Yuster.)

Example
$X = \{A \in T^{n\times n} \mid \det(A) = 0\}$
Each $I \subseteq [n] \times [n]$ of cardinality $n^2 - 1$ satisfies the requirement: $T^{(n-1)\times(n-1)} \times (T^{(n-1)})^2 \to T^{n\times n}$,
$(A, v, w) \mapsto \begin{bmatrix} A & v \\ w^t & w^t A^{-1} v \end{bmatrix}$
Hence X is tropically unirational.

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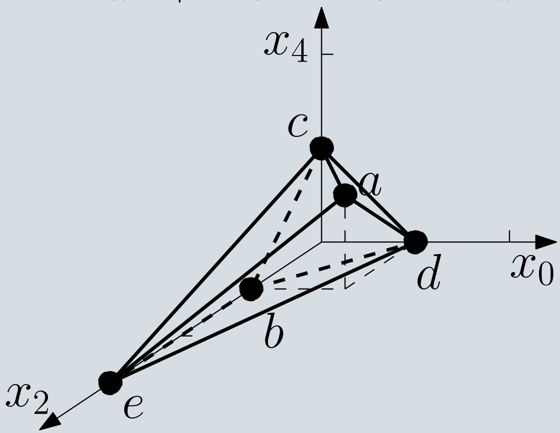
…and one more

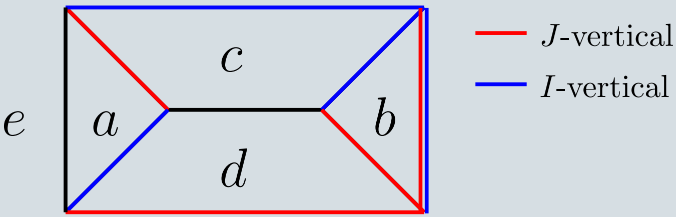
(suggested by Cools and Sturmfels)
$Y \subseteq K^5$ parameterised by $(s^4, s^3t, \ldots, t^4)$ (cone over rational normal quartic)
$X = \overline{Y + Y}$ first secant variety, zero set of

$$\det \begin{bmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{bmatrix}$$

$$= z_0z_2z_4 + 2z_1z_2z_3 - z_1^2z_4 - z_0z_3^2 - z_2^3$$
$$= a + 2b - c - d - e$$



Secant variety, continued



$I = \{1,2,3,4\}$ and $J = \{0,1,2,3\}$ satisfy $\pi_I, \pi_J : X \dashrightarrow T^4$ birational.

Two suitable reparameterisations of the parameterisation $(s_1^4 + s_2^4, \ldots, t_1^4 + t_2^4)$ tropicalise to maps that together cover the cone where $b = e \leq a, c, d$. Hence X is tropically unirational.

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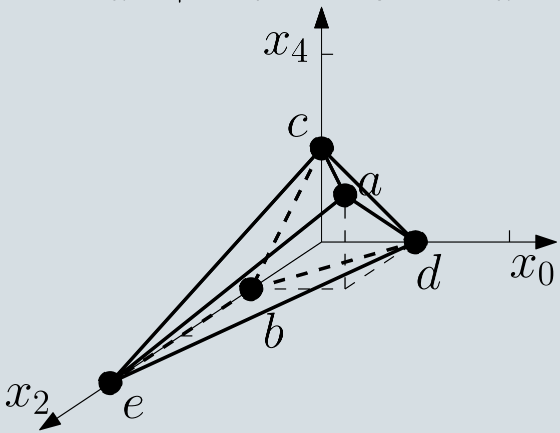

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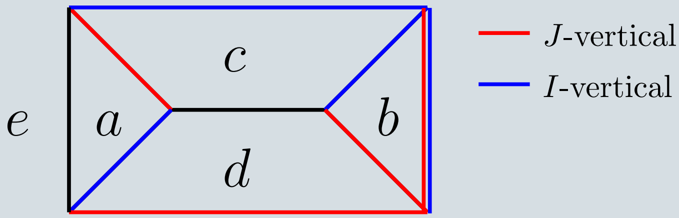

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Two loose ends

Observation
$\varphi : T^m \dashrightarrow T^n$, $X = \overline{\operatorname{im}\varphi}$, $\dim X = d$, $\pi : T^n \to T^{d+1}$ generic homomorphism
Then φ has a tropically surjective reparameterisation iff $\pi \circ \varphi$ does.

Theorem
$\operatorname{char} K = 0$
$X \subseteq T^n$ unirational
$\operatorname{Trop}(X) = P_1 \cup \ldots \cup P_N$
P_i $v(K^*)$-rational relatively open polyhedra
Then X has a parameterisation ψ such that $\operatorname{im}\operatorname{Trop}(\psi)$ contains open subsets of all P_i.

(Exercise in valuations.)

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