Tropical reparameterisations

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Two ways to describe a line

implicitly, by equations

$X := \{(x, y) \mid y - x - 1 = 0\} \subset \mathbb{A}^2$

explicitly, by parameterisation

$\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2, \quad u \mapsto (u, u + 1); \quad X = \operatorname{im} \phi$

elimination theory: parameterisation $\rightsquigarrow$ equations?

Tropicalising those two ways

by equations

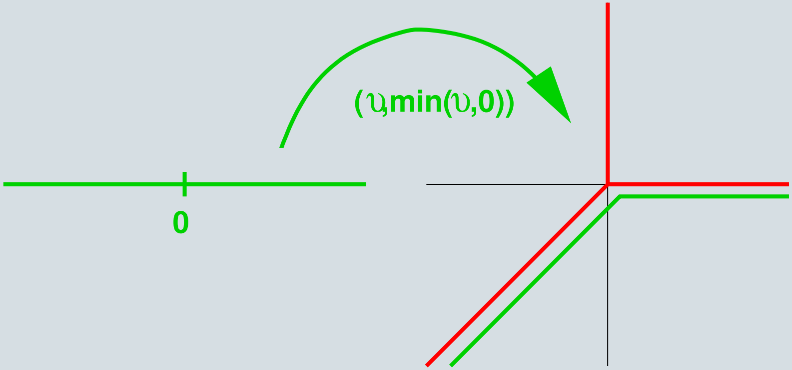
$X = \{(x,y) \mid y - x - 1 = 0\} \subset \mathbb{A}^2$

$\mathcal{T}X = \{(\xi,\eta) \mid \min\{\eta,\xi,0\}$ is attained at least twice$\} \subset {\mathbb{R}_\infty}^2$

by parameterisation

$\phi : u \mapsto (u, u+1)$

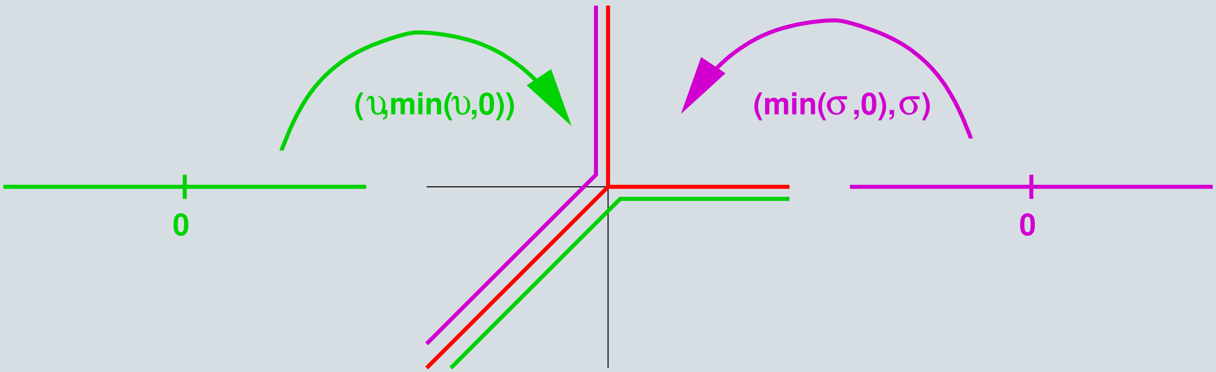
$\mathcal{T}\phi : v \mapsto (v, \min\{v, 0\})$



$\operatorname{im} \mathcal{T}\phi \subseteq \mathcal{T}X$, in general $\subsetneq$

Reparameterisation for the line

$$\alpha : \mathbb{A}^1 \longrightarrow \mathbb{A}^1, \quad s \mapsto s - 1$$
$$\phi' := \phi \circ \alpha : \mathbb{A}^1 \longrightarrow \mathbb{A}^2, \quad s \mapsto (s - 1, s)$$
$$\mathcal{T}(\phi') : \mathbb{R}_\infty \longrightarrow {\mathbb{R}_\infty}^2, \quad \sigma \mapsto (\min\{\sigma, 0\}, \sigma)$$



Two reparameterisations tropically cover $\mathcal{T}X$

Four questions

$\phi : \mathbb{A}^m \longrightarrow \mathbb{A}^n$ polynomial map
$X := \overline{\operatorname{im} \phi}$ algebraic variety
then $\operatorname{im} \mathcal{T}\phi \subseteq \mathcal{T}X$

$\exists$? finitely many (or one) reparameterisations $\alpha_i : \mathbb{A}^{p_i} \longrightarrow \mathbb{A}^m$ (or rational maps) such that $\cup_i \operatorname{im} \mathcal{T}(\phi \circ \alpha_i) = \mathcal{T}(X)$.

Remark. Sturmfels-Tevelev-Yu (2007) describe $\mathcal{T}X$ from ϕ in case of generic coefficients; generalisations use Hacking-Keel-Tevelev's *geometric tropicalisation* (2007).

Two observations

Lemma. *If $\phi = (\phi_1, \dots, \phi_n)$ with all ϕ_i homogeneous of same degree, then the four questions are equivalent.*

Multiply with common denominator;
combine several reparameterisations into one.

Proposition. *All four questions reduce to the case where X is a hypersurface in $\mathbb{A}^n$.*

Choose "generic" mononomial map $\pi : \mathbb{A}^n \to \mathbb{A}^{d+1}$ where $d = \dim X$;
reparameterisations that work for $\pi \circ \phi$ also work for ϕ.

Linear spaces

Theorem (Yu-Yuster, 2006). $\phi : \mathbb{A}^m \to X \subseteq \mathbb{A}^n$ linear, *given by a matrix* ϕ *Then* $\operatorname{im} \mathcal{T}\phi = \mathcal{T}X$ *iff every vector* $v \in X$ *of minimal support* (cocircuit) *is scalar multiple of a column of* ϕ.

(This can be achieved form by composing ϕ with a linear map $\mathbb{A}^p \to \mathbb{A}^m$.)

Example. $\phi : \mathbb{A}^2 \to \mathbb{A}^3$ given by $\phi = \begin{bmatrix} t & 0 \\ 0 & 1 \\ 1 & t \end{bmatrix}$ over $\mathbb{C}((t))$

$X = \{(x, y, z)^T \mid x + t^2 y - tz = 0\}$
$\mathcal{T}X = C_1 \cup C_2 \cup C_3$ with
$C_1 = \{(\xi, \xi - 2, \zeta) \mid \zeta \geq \xi - 1\}$
$C_2 = \{(\xi, \eta, \xi - 1) \mid \eta \geq \xi - 2\}$
$C_3 = \{(\xi, \eta, \eta + 1) \mid \xi \geq \eta + 2\}$

$\mathcal{T}\phi : (\alpha, \beta) \mapsto (\alpha + 1, \beta, \min\{\alpha, \beta + 1\})$; $\operatorname{im} \mathcal{T}\phi = C_2 \cup C_3$

Example, continued

$$\phi \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} t & 0 \\ 0 & 1 \\ 1 & t \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} t & 0 & t^2 \\ 0 & 1 & -1 \\ 1 & t & 0 \end{bmatrix}$$

The last matrix contains all cocircuits of X, so

$$\operatorname{im} \mathcal{T} \left(\phi \circ \begin{bmatrix} 1 & 0 & t \\ 0 & 1 & -1 \end{bmatrix} \right) = C_1 \cup C_2 \cup C_3 = \mathcal{T} X$$

by Yu and Yuster's theorem.

A Grassmannian from a linear space

$\phi : \mathbb{A}^n \to X \subseteq \mathbb{A}^{\binom{n}{2}}, \quad (x_1, \ldots, x_n) \mapsto (x_i - x_j)_{i<j}$
zero patterns in the image $\leftrightsquigarrow$ partitions of $\{1, \ldots, n\}$
cocircuits $\leftrightsquigarrow$ partitions into two parts
so $\exists \alpha : \mathbb{A}^{2^{n-1}-1} \to \mathbb{A}^n$ linear with $\operatorname{im} \mathcal{T}(\phi \circ \alpha) = \mathcal{T} X$

$\psi : \mathbb{A}^n \times \mathbb{A}^n \to Y \subseteq \mathbb{A}^{\binom{n}{2}}, \quad (u, x) \mapsto (u_i u_j (x_i - x_j))_{i<j}$

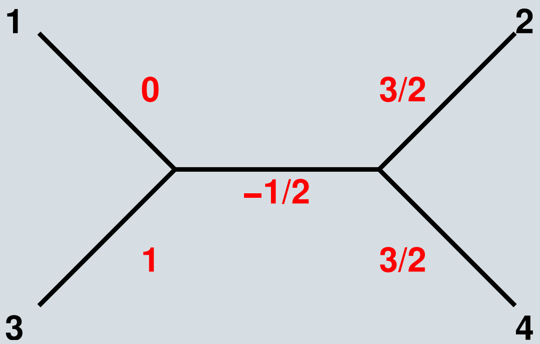
Y = Grassmannian of 2-dimensional subspaces of n-space
$\operatorname{im} \mathcal{T}(\psi \circ (\operatorname{id} \times \alpha)) = \mathcal{T} Y$, the *tropical Grassmannian*
studied by Speyer and Sturmfels (2004) and many others

Points of Y correspond to *tree metrics*, by the above obtained
from tropical linear combinations of 2-partitions by stretching ends.

Example with $n = 4$

$\{1, 234\}$ short-hand for $(d_{ij})_{i<j}$ with $d_{1j} = 0$ and all other $d_{ij} = \infty$

$(1 \otimes \{1, 234\}) \oplus (2 \otimes \{13, 24\}) \oplus (3 \otimes \{14, 23\}\})$
equals the tree metric of



Remark. Internal edges have negative length.
Edges adjacent to leaves can be arbitrarily altered using the v_i.

Local tropical reparameterisations

$\phi : \mathbb{A}^m \longrightarrow \mathbb{A}^n$ polynomial map
$X := \overline{\operatorname{im} \phi}$ algebraic variety of dimension d

Theorem. *For almost all* $\xi \in \mathcal{T}X$
$\exists \alpha : \mathbb{T}^d \longrightarrow \mathbb{A}^m$ *such that*
$\operatorname{im} \mathcal{T}(\phi \circ \alpha) \supset$ *a* d*-dimensional neighbourhood of* ξ.

Remark.

- α is allowed to have Laurent polynomial components
- d is also the dimension of $\mathcal{T}X$
- if all ϕ_i homogeneous of the same degree, k such local reparameterisations can be combined to a reparameterisation $\mathbb{A}^{kd} \longrightarrow \mathbb{A}^m$
- *almost all* means the ξ_i span a d-dimensional $\mathbb{Q}$-subspace of $\mathbb{R}$

Proof sketch

1. assume $\xi_1, \ldots, \xi_d$ linearly independent over $\mathbb{Q}$
2. consider $K = \mathbb{C}(t_1, \ldots, t_d)$ with valuation $v(t_i) = \xi_i$
3. take a point p of $\mathbb{A}^m$ with coordinates in $\overline{\widehat{K}}$ such that $v(\phi(p)) = \xi$; exists
4. approximate p with a point q in $\mathbb{C}[t_1^{\pm 1/N}, \ldots, t_d^{\pm 1/N}]$ such that $v(\phi(q)) = \xi$ (multivariate Puiseux theorem)
5. set $u_i := t_i^{1/N}$
6. $q(u_1, \ldots, u_n)$ is the required reparameterisation

Remark.
- not yet very constructive, but I'm collaborating with Anders Jenssen to make it so
- not clear that finitely many suffice...