A tropical approach to secant dimensions

Jan Draisma

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Typical example: polynomial interpolation in two variables

Set–up:
$d \in \mathbb{N}$
$p_1, \ldots, p_k$ general points in $\mathbb{C}^2$
$\operatorname{codim}\{f \in \mathbb{C}[x,y]_{\leq d} \mid \forall i : f(p_i) = f_x(p_i) = f_y(p_i) = 0\} = ??$
expect: $\min\{3k, \binom{d+2}{2}\}$ (upper bound)

Hirschowitz (1985):
correct, unless $(d,k) = (2,2)$ or $(d,k) = (4,5)$ (1 instead of 0)

D (2006): new proof using tropical geometry, paper and scissors
Alexander and Hirschowitz (1995): more variables
Also doable tropically??
Brannetti (2007, student of Ciliberto): three variables, tropically.

Tropical geometry

Set–up:
K field
$v : K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ non–Archimedean valuation,
that is, $v^{-1}(\infty) = \{0\}$, $v(ab) = v(a) \odot v(b)$, and $v(a+b) \geq v(a) \oplus v(b)$
e.g. K =Laurent series and v=multiplicity of 0 as a zero
technical conditions on (K, v)

$X \subseteq K^n$ closed subvariety
$\rightsquigarrow \mathcal{T}X := \{v(x) = (v(x_1), \ldots, v(x_n)) \mid x \in X\}$
tropicalisation of X

depends on coordinates!

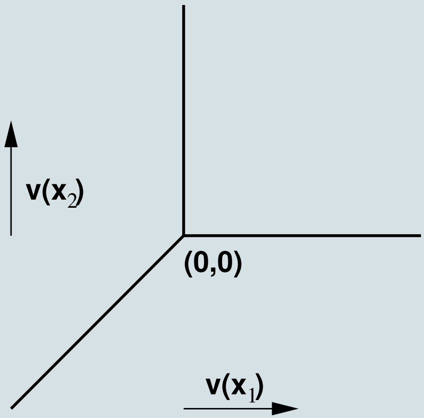
Codimension one varieties

X zero set of one polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$
$\mathcal{T}f(\xi) := \min_{\alpha \in \mathbb{N}^n}(v(c_\alpha) + \langle \xi, \alpha \rangle)$ tropicalisation of f

Theorem 1 (Einsiedler–Kapranov–Lind).
$\mathcal{T}X = \{\xi \in \overline{\mathbb{R}}^n \mid \mathcal{T}f \text{ not linear at } \xi\}$

$\rightsquigarrow$ tropical hypersurfaces are polyhedral complexes!

Example:
$f = x_1 + x_2 - 1$ (line)
$\mathcal{T}f = \min\{\xi_1, \xi_2, 0\}$



Arbitrary codimension

I the ideal of $X \subseteq K^n$

Theorem 2 (EKL 2004, SS 2003, see also D 2006).

$$\begin{aligned} \mathcal{T}X &= \{(v'(x_1), \ldots, v'(x_n)) \mid v' : K[X] \to \overline{\mathbb{R}} \textit{ ring valuation extending } v\} \\ &= \{w \in \overline{\mathbb{R}}^n \mid \forall f \in I : \mathcal{T}f \textit{ not linear at } w\} \end{aligned}$$

Theorem 3 (Bogart–Jensen–Speyer–Sturmfels–Thomas (2005)).
$\exists$ *finite subset of* I *for which previous theorem is true*

$\rightsquigarrow$ *tropical basis* (hard to compute!)
$\rightsquigarrow$ $\mathcal{T}X$ is a polyhedral complex

Theorem 4 (Bieri–Groves (1985), Sturmfels).
X *irreducible of dimension* $d \Rightarrow \dim \mathcal{T}X = d$

Example: $\mathcal{T}\mathrm{SL}_n$ and $\mathcal{T}\mathrm{O}_n$

Codimension one:
$\mathcal{T}\mathrm{SL}_n = \{A \in \overline{\mathbb{R}}^{n\times n} \mid \operatorname{tdet} A \leq 0$ and if < 0 then attained at least twice$\}$
monoid under tropical matrix multiplication

Higher codimension:
$\mathrm{O}_n := \{g \mid g^t g = I\}$
$\mathcal{T}\mathrm{O}_n =??$
closed under tropical matrix multiplication? probably so
tropical basis? not sufficient to tropicalise the n^2 defining equations

Proposition 5 (D-McAllister 2006). $\mathcal{T}\mathrm{O}_n \supset \{$ *semi-metrics* (d_{ij}) *on* n *points*$\}$ *(full-dimensional cone)*

Corollary 6. *"Composition" of metrics stays within an* $\binom{n}{2}$*-dimensional complex!*

Secant varieties

C a closed cone in a K-space V, $k \in \mathbb{N}$
$kC := \overline{\{v_1 + \ldots + v_k \mid v_i \in C\}}$,
the k-th *secant variety* of C
Main reference: Zak, *Tangents and secants of algebraic varieties*, 1993.

Example 7.

- $C_1 =$ rank ≤ 1 matrices in $V_1 = M_m$
  $kC_1 =$ rank $\leq k$ matrices
- $C_2 = \{z_1 \wedge z_2\} \subseteq V_2 := \bigwedge^2 K^m$
  cone over Grassmannian of 2-spaces in K^m
  $kC_2 =$ skew-symmetric matrices of rank $\leq 2k$
- $C_3 =$ cone over Grassmannian of *isotropic* 2-spaces in K^m
  $2C_3$ and $3C_3$ are complicated
  $kC_3 = kC_2$ for $k \geq 4$ (Baur and Draisma, 2004)

Non-defectiveness

Note: $\dim kC \leq \min\{k \dim C, \dim V\}$, the *expected dimension*.

Definition 8.
kC is *non-defective* if $\dim kC$ is as expected.
C is non-defective if all kC are.

Many Cs are non-defective, but hard to prove so!

Minimal orbits: interesting cones

V irrep of complex algebraic group G
$v \in V$ highest weight vector
$C := Gv \cup \{0\} \subseteq V$

all examples so far were of this form
dimensions of $kC, \; k = 1, 2, \ldots$ largely unknown!
(except $V = S^d(\mathbb{C}^n)$ for $G = \mathrm{GL}_n$—Alexander & Hirschowitz (1995))

widely open: tensor products, Grassmannians, etc.

Tropical strategy for proving kC non-defective

Recall :

algebraic geometry		tropical geometry
embedded affine variety $X \subseteq K^n$	$\longrightarrow$	polyhedral complex $\mathcal{T}(X) \subseteq \overline{\mathbb{R}}^n$
polynomial map f	$\longrightarrow$	piecewise linear map $\mathcal{T}(f)$
$\dim X$	$=$	$\dim \mathcal{T}(X)$

Strategy: prove $\dim \mathcal{T}(kC) = k \dim C$; then kC is non-defective.
But kC not known, let alone $\mathcal{T}(kC)$!
Proposal:

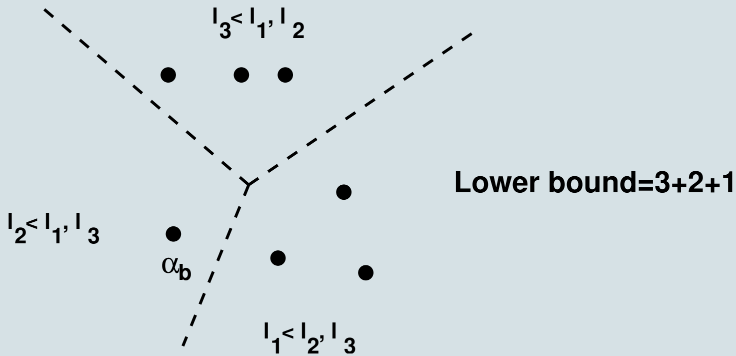
- parameterise $h : K^m \to C \subseteq V$
- tropicalise $f : (K^m)^k \to kC,\ (z_1, \ldots, z_k) \mapsto h(z_1) + \ldots + h(z_k)$
- compute $\operatorname{rk} dT(f)$ at a good point $\rightsquigarrow$ lower bound on $\dim \mathcal{T} kC$

A (simplified) theorem

$h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n$ parameterisation
assume each $h_b = c_b x^{\alpha_b} \neq 0$ (1 term)

for $l = (l_1, \ldots, l_k)$ k affine-linear functions on $\mathbb{R}^m$ set
$C_i(l) := \{\alpha_b | l_i(\alpha) < l_j(\alpha) \text{ for all } j \neq i\}$

Theorem 9 (Draisma, 2006). $\dim kC \geq \sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} C_i(l))$



Funny optimisation problem

$A \subseteq \mathbb{R}^n$ finite, $k \in \mathbb{N}$
Maximise $\sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} C_i(l))$=:*
over all $l = (l_1, \ldots, l_k)$, each l_i affine-linear

Corollary 10.
$A = \{\alpha_b \mid b\}$ *exponents of monomials in parameterisation*
draw A *on* m*-dimensional paper*
cut paper into k *pieces*
compute sg. like $*$
$\rightsquigarrow$ *lower bound on* $\dim kC$

Example from beginning:

C =Veronese cone
$h : (x_1, x_2, x_3) \mapsto (x_1e_1 + x_2e_2 + x_3e_3)^d \in S^d(\mathbb{C}^3)$
satisfies conditions of theorem $\rightsquigarrow$ paper-and-scissors lower bound

Generalisation to higher dimensions?

Some results (with Karin Baur)

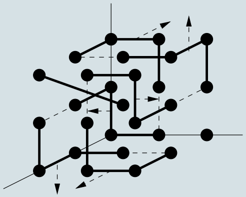
Non-degenerate:

1. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 2)$
2. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 1, 1)$
3. Segre embedding of $(\mathbb{P}^1)^6$ (cells for $k = 9$: 8 disjoint Hamming balls of radius 1 and one cell in the middle)
4. $\{$flags point $\subseteq$ line $\subseteq$ $\mathbb{P}^2\}$ probably all non-defective except those of highest weight $\omega_1 + \omega_2$ (adjoint representation) or $2\omega_1 + 2\omega_2$ **needs generalisation of Theorem 9**

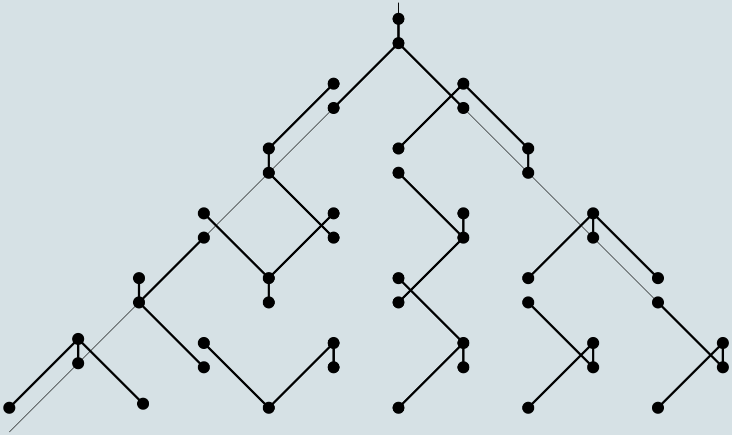
Almost done: $\mathbb{P}^1 \times \mathbb{P}^2$

Conjecture 11. *The lower bound always gives correct dimension for Segre-Veronese embeddings.*

Some pictures



picture suggests: S-V embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2, 2, 2)$ has defective $7C$. Indeed!



Minimal orbit in representation of SL_3 of highest weight $(5, 1)$ is non-defective.

Conclusions

Non-defectiveness often provable by optimising a strange polyhedral-combinatoric objective function

Hope a point in $T(kC)$ with full-dimensional neighbourhood gives restrictions on the ideal of kC. Sufficient to settle one or two more cases of GSS?

Segre-Veronese is the given bound always correct?

Other minimal orbits Smallest flag variety has non-monomial parameterisation, but doable with a trick. In general: which parameterisation to use? (Littelmann-Bernstein-Zelevinsky polytopes?)

Tropical geometry is a powerful tool! (and interesting in its own right..)