A tropical approach to secant varieties

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Prototypical example: polynomial interpolation

Set–up:
$d \in \mathbb{N}$
$p_1, \ldots, p_k$ general points in $\mathbb{C}^2$
$\operatorname{codim}\{f \in \mathbb{C}[x,y]_{\leq d} \mid \forall i : f(p_i) = f_x(p_i) = f_y(p_i) = 0\} = ??$
expect: $\min\{3k, \binom{d+2}{2}\}$ (upper bound)

Hirschowitz (1985):
correct, unless $(d,k) = (2,2)$ or $(d,k) = (4,5)$ ($\dim 1$ instead of 0)

D (2006): new proof using tropical geometry, paper and sciccors

Alexander and Hirschowitz (1995): more variables
Also doable tropically??

Brannetti (2007, student of Ciliberto): three variables, tropically

Secant varieties

C a closed cone in a K-space V, $k \in \mathbb{N}$
$kC := \overline{\{v_1 + \ldots + v_k \mid v_i \in C\}}$,
the k-th *secant variety* of C
Main reference: Zak, *Tangents and secants of algebraic varieties*, 1993.

Example 1.

- $C_1 =$ rank ≤ 1 matrices in $V_1 = M_m$
  $kC_1 =$ rank $\leq k$ matrices
- $C_2 = \{z_1 \wedge z_2\} \subseteq V_2 := \bigwedge^2 K^m$
  cone over Grassmannian of 2-spaces in K^m
  $kC_2 =$ skew-symmetric matrices of rank $\leq 2k$
- $C_3 =$ cone over Grassmannian of *isotropic* 2-spaces in K^m
  $2C_3$ and $3C_3$ are complicated
  $kC_3 = kC_2$ for $k \geq 4$ (Baur and Draisma, 2004)

Non-defectiveness

Note: $\dim kC \leq \min\{k \dim C, \dim V\}$, the *expected dimension*.

Definition 2.
kC is *non-defective* if $\dim kC$ is as expected.
C is non-defective if all kC are.

Many Cs are non-defective, but hard to prove so!

Secant dimensions known for:

- Veronese embeddings (Alexander-Hirschowitz)
- certain Grassmannians and certain Segre(-Veronese) embeddings (Catalisano, Geramita, Gimigliano)
- certain highest-weight orbits (Baur, Draisma, de Graaf)

Goal of this talk

Combinatorial lower bound for $\dim kC$ where

$$C = \{v_1^{d_1} \otimes v_2^{d_2} \otimes \ldots \otimes v_p^{d_p} \mid v_i \in \mathbb{C}^{n_i}\} \subseteq S^{d_1}(\mathbb{C}^{n_1}) \otimes \ldots \otimes S^{d_p}(\mathbb{C}^{n_p})$$

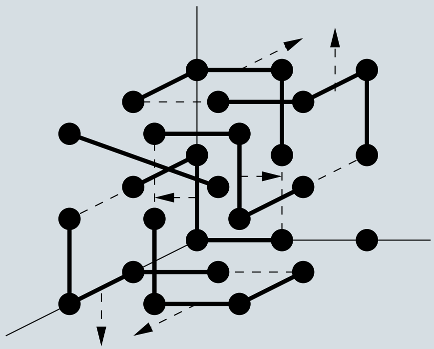
(Segre-Veronese embeddings)

Conjecture 3. *This lower bound is always sharp.*

Lots of evidence!

Example 4.
$S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^2) \otimes S^2(\mathbb{C}^2)$:

$(\dim kC)_k$
$= (4, 8, 12, 16, 20, 24, 26, 27)$



Aside: relation to polynomial interpolalation

$V = K^m$
$C := \{v^d\} \subseteq S^d V$
$\dim kC = ?$

Lemma 5 (Terracini, 1911). *For* $v_1, \ldots, v_k \in V$ *generic*
$\dim kC = \dim T_{v_1^d} C + \ldots + \dim T_{v_k^d} C.$

Lasker, 1904:
$T_{v_i^d} C = \{f \in S^d(V^*) \mid f \text{ is singular in } [v_i] \in \mathbb{P}V\}^0$
so
$\dim kC = \operatorname{codim} \{\text{hom. pols. of degree } d \text{ singular in all } [v_i]\}.$

Tropical geometry: main definition

Set–up:
$v : K \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ non–Archimedean valuation

($v^{-1}(\infty) = \{0\}$, $v(ab) = v(a) + v(b)$, and $v(a+b) \geq \min\{v(a), v(b)\}$
think K =Laurent series over $\mathbb{C}$ in t)

technical conditions on (K, v)

$X \subseteq K^n$ closed subvariety
$\rightsquigarrow \mathcal{T}X := \{v(x) = (v(x_1), \ldots, v(x_n)) \mid x \in X\}$
tropicalisation of X

depends on coordinates!

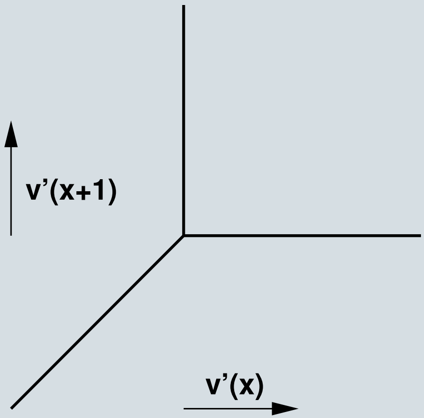
Codimension one

X zero set of one polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$
$\mathcal{T}f(\xi) := \min_{\alpha \in \mathbb{N}^n}(v(c_\alpha) + \langle \xi, \alpha \rangle)$ tropicalisation of f

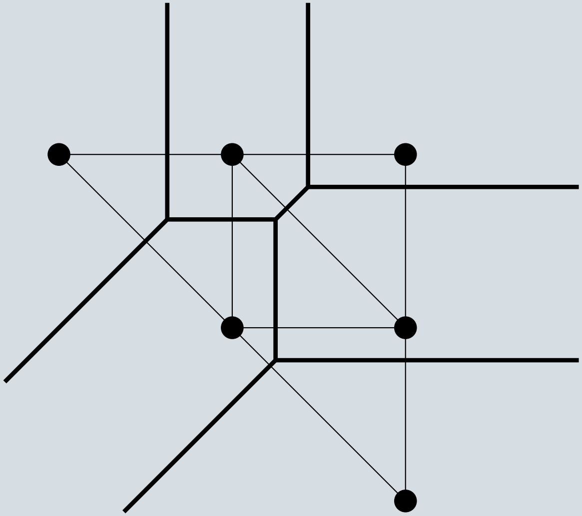
Theorem 6 (Einsiedler–Kapranov–Lind).
$\mathcal{T}X = \{\xi \in \overline{\mathbb{R}}^n \mid \mathcal{T}f$ *not linear at* $\xi\}$

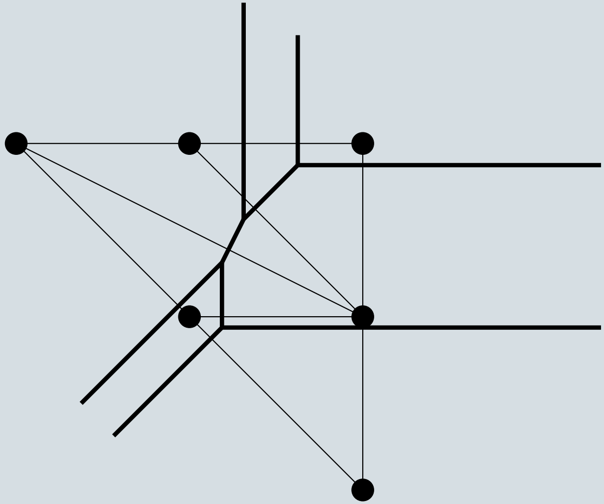
$\rightsquigarrow$ tropical hypersurfaces are polyhedral complexes!

Example:
$f = x_1 + x_2 - 1$ (line)
$\mathcal{T}f = \min\{\xi_1, \xi_2, 0\}$

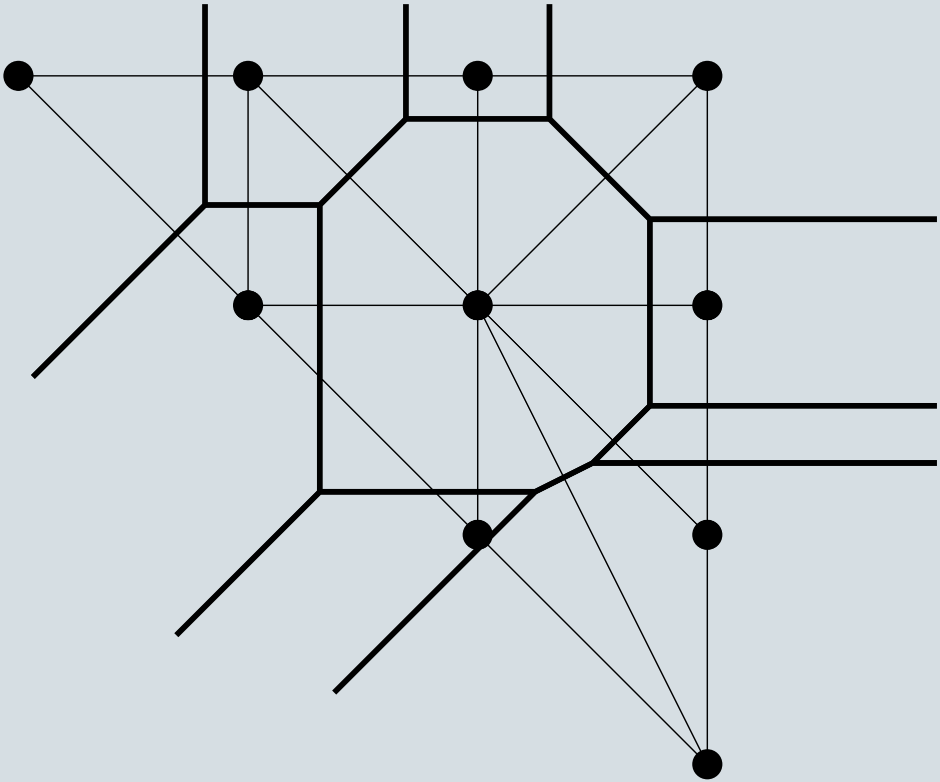


Plane Curves: conics





Plane Curves: a cubic



Aside: counting plane curves

Proposition 7. $\exists$ *characterisation of tropical curves of degree d in the plane.*

Mikhalkin (re)computed the number of *classical* degree d, genus g plane curves through $3d + g - 1$ general points:
count *tropical* such curves, each with a certain multiplicity.

Caporaso–Harris in 1998 needed havier algebraic geometry!

algorithms for enumerating such tropical curves
(Mikhalkin, Gathmann–Markwig)

Higher codimension

I the ideal of $X \subseteq K^n$

Theorem 8 (EKL 2004, SS 2003, see also D 2006).

$$\begin{aligned} \mathcal{T}X &= \{(v'(x_1), \ldots, v'(x_n)) \mid v' : K[X] \to \overline{\mathbb{R}} \textit{ ring valuation extending } v\} \\ &= \{w \in \overline{\mathbb{R}}^n \mid \forall f \in I : \mathcal{T}f \textit{ not linear at } w\} \end{aligned}$$

Theorem 9 (Bogart–Jensen–Speyer–Sturmfels–Thomas (2005)).
$\exists$ *finite subset of* I *for which previous theorem is true*

$\rightsquigarrow$ *tropical basis* (hard to compute!)
$\rightsquigarrow$ $\mathcal{T}X$ is a polyhedral complex

Theorem 10 (Bieri–Groves (1985), Sturmfels).

X *irreducible of dimension* $d \Rightarrow \dim \mathcal{T}X = d$

Tropical lower bounds on $\dim kC$

algebraic geometry		tropical geometry
embedded affine variety $X \subseteq K^n$	$\rightarrow$	polyhedral complex $\mathcal{T}(X) \subseteq \overline{\mathbb{R}}^n$
polynomial map f	$\rightarrow$	piecewise linear map $\mathcal{T}(f)$
$\dim X$	$=$	$\dim \mathcal{T}(X)$

Strategy: prove $\dim \mathcal{T}(kC) = k \dim C$; then kC is non–defective.
But kC not known, let alone $\mathcal{T}(kC)$!
Proposal:

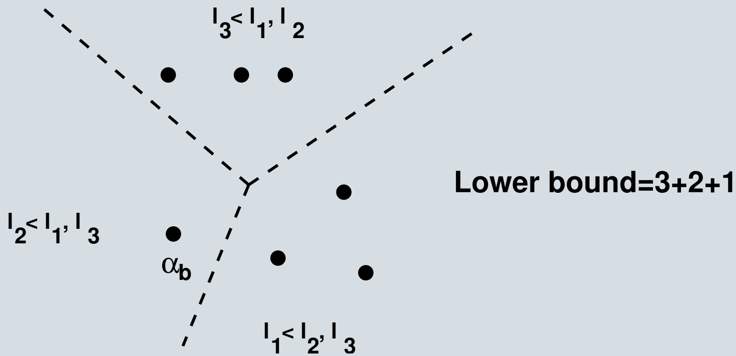
- parameterise $h : K^m \rightarrow C \subseteq V$
- tropicalise $f : (K^m)^k \rightarrow kC,\ (z_1, \ldots, z_k) \mapsto h(z_1) + \ldots + h(z_k)$
- compute $\operatorname{rk} dT(f)$ at a good point $\rightsquigarrow$ lower bound on $\dim \mathcal{T} kC$

A (simplified) theorem

$h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n$ parameterisation
assume each $h_b = c_b x^{\alpha_b} \neq 0$ (1 term)

for $l = (l_1, \ldots, l_k)$ k affine–linear functions on $\mathbb{R}^m$ set
$C_i(l) := \{\alpha_b | l_i(\alpha_b) < l_j(\alpha_b) \text{ for all } j \neq i\}$

Theorem 11 (Draisma, 2006). $\dim kC \geq \sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} C_i(l))$



Funny optimisation problem

$A \subseteq \mathbb{R}^n$ finite, $k \in \mathbb{N}$
Maximise $\sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} C_i(l))$=:*
over all $l = (l_1, \ldots, l_k)$, each l_i affine–linear

Corollary 12.
$A = \{\alpha_b \mid b\}$ *exponents of monomials in parameterisation*
draw A *on* m*–dimensional paper*
cut paper into k *pieces*
compute sg. like $*$
$\rightsquigarrow$ *lower bound on* $\dim kC$

Generalisation of Theorem 11

Optimisation problem:
Given
$k \in \mathbb{N}$
$A = (A_1, \ldots, A_n)$ list of finite subsets of $\mathbb{R}^n$
Optimisation domain
k-tuples $l = (l_1, \ldots, l_k)$ of affine linear functions on $\mathbb{R}^n$
Objective function
$\sum_{i=1}^{k}(1 + \dim \operatorname{Aff}_{\mathbb{R}} C_i(l))$
where $C_i = \bigcup_{b=1}^{n}\{\alpha \in A_b \mid f_i(\alpha) < f_j(\beta) \text{ for all } (\beta, j) \in A_b \times \{1, \ldots, k\}\}$
Optimal value $\mathrm{AP}^*(A, k)$

Theorem 13. *$h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n$ parametrisation*
$A_b \subseteq \mathbb{N}^m$ support of h_b
then $\mathrm{AP}^*(A, k) \leq \dim kC$

Results with Karin Baur

Non–degenerate:

- all Segre–Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 2)$ (Catalisano-Geramita-Gimigliano)
- all Segre–Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 1, 1)$ (Catalisano-Geramita-Gimigliano)
- Segre embedding of $(\mathbb{P}^1)^6$ (cells for $k = 9$: 8 disjoint Hamming balls of radius 1 and one cell in the middle)
- $\{\text{flags point} \subseteq \text{line} \subseteq \mathbb{P}^2\}$ non–defective in all embeddings except those of highest weight $\omega_1 + \omega_2$ (adjoint representation) or $2\omega_1 + 2\omega_2$
- all Segre–Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^2$ except $(2, \text{even})$ and $(3, 1)$

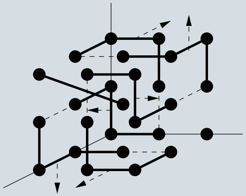
Rationale

Results by others:

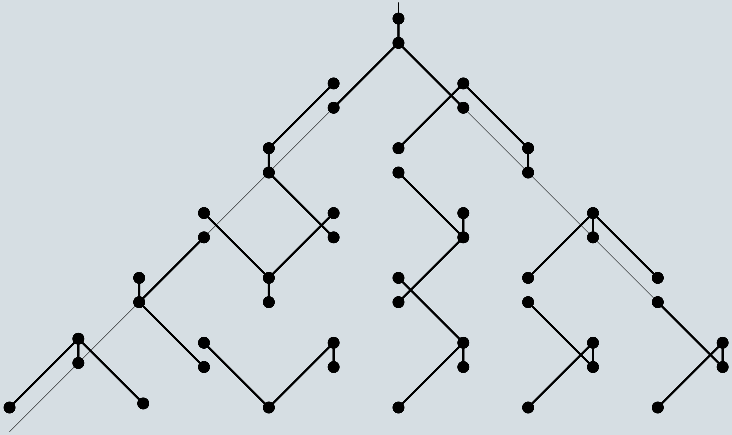
- Veronese embeddings of $\mathbb{P}^3$ (Brannetti)
- Segre embeddings of $(\mathbb{P}^1)^d$ for $d = 1, \ldots, 9$ (Halupczok, tropically with computer)

Conjecture: tropical lower bound sharp for all Segre-Veronese embeddings.

Some pictures



picture suggests: S-V embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2,2,2)$ has defective $7C$. Indeed!



Minimal orbit in representation of SL_3 of highest weight $(5,1)$ is non-defective.

Conclusion

Non-defectiveness often provable by optimising a strange polyhedral-combinatoric objective function

Hope a point in $T(kC)$ with full-dimensional neighbourhood gives restrictions on the ideal of kC. Sufficient to settle one or two more cases of GSS?

Segre-Veronese is the given bound always correct?

Other minimal orbits Smallest flag variety doable with a trick: reduce all A_b in Theorem 13 to singletons, and use Voronoi-variant. In general: which parametrisation to use? (Littelmann-Bernstein-Zelevinsky polytopes?)