A tropical approach to secant varieties

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Typical example: polynomial interpolation in two variables

Set–up:
$d \in \mathbb{N}$
$p_1, \ldots, p_k$ general points in $\mathbb{C}^2$
$\mathrm{codim}\{f \in \mathbb{C}[x,y]_{\leq d} \mid \forall i : f(p_i) = f_x(p_i) = f_y(p_i) = 0\} =??$
expect: $\min\{3k, \binom{d+2}{2}\}$ (upper bound)

Hirschowitz (1985):
correct, unless $(d,k) = (2,2)$ or $(d,k) = (4,5)$ (1 instead of 0)

D (2006): new proof using tropical geometry, paper and scissors

Alexander and Hirschowitz: more variables (1995)
Also doable tropically??

Tropical Semiring

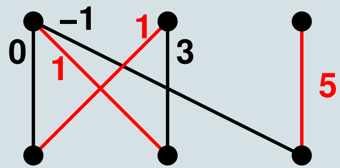
$\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$
$a \oplus b := \min\{a, b\}$
$a \odot b := a + b$

$\infty \oplus b = b$
$0 \odot b = b$
$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

Tropical polynomials

$\alpha, \beta, \ldots \in \mathbb{N}^n$
$a, b, \ldots \in \overline{\mathbb{R}}$
the map $\overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$
$\xi \mapsto a \odot \bigodot_{i=1}^n \xi_i^{\odot \alpha_i} \oplus b \odot \bigodot_{i=1}^n \xi_i^{\odot \beta_i}$
$= \min\{a + \langle \xi, \alpha \rangle, b + \langle \xi, \beta \rangle, \ldots\}$ is a *tropical polynomial*

Example
$A \in \overline{\mathbb{R}}^{n \times n}$
$\mathrm{tdet}(A) := \bigoplus_{\pi \in S_n} a_{\pi(1),1} \odot a_{\pi(2),2} \odot \cdots \odot a_{\pi(n),n}$ *tropical determinant*
minimal weight matching in $K_{n,n}$ with edge weights a_{ij}



tdet = 1+5+5

Tropical geometry

Set–up:
K field
$v : K \to \overline{\mathbb{R}}$ non–Archimedean valuation,
that is, $v^{-1}(\infty) = \{0\}$, $v(ab) = v(a) \odot v(b)$, and $v(a+b) \geq v(a) \oplus v(b)$
e.g. K =Laurent series and v=multiplicity of 0 as a zero
technical conditions on (K, v)

$X \subseteq K^n$ given by polynomial equations
$\rightsquigarrow \mathcal{T}X := \{v(x) = (v(x_1), \ldots, v(x_n)) \mid x \in X\}$
tropicalisation of X

depends on coordinates!

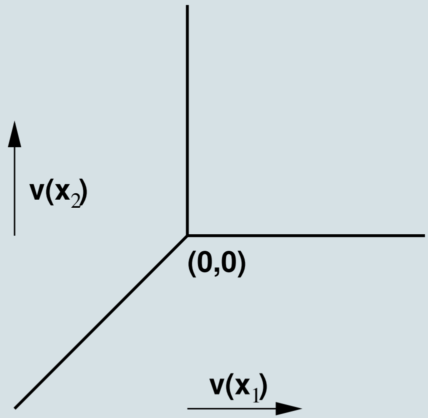
Codimension one varieties

X zero set of one polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$
$\mathcal{T}f(\xi) := \min_{\alpha \in \mathbb{N}^n}(v(c_\alpha) + \langle \xi, \alpha \rangle)$ tropicalisation of f

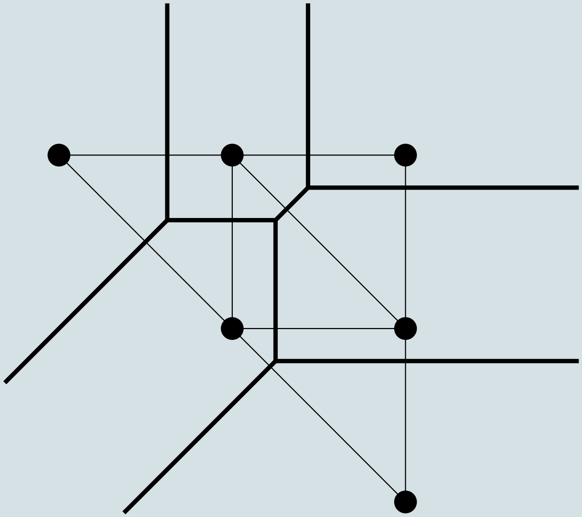
Theorem 1 (Einsiedler–Kapranov–Lind).
$\mathcal{T}X = \{\xi \in \overline{\mathbb{R}}^n \mid \mathcal{T}f$ *not linear at* $\xi\}$

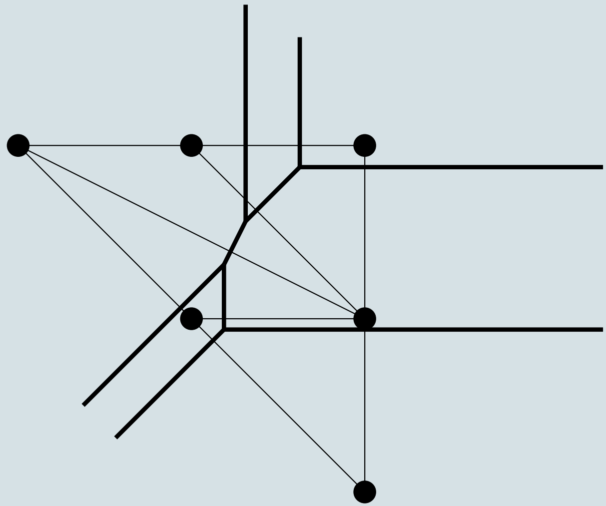
$\rightsquigarrow$ tropical hypersurfaces are polyhedral complexes!

Example:
$f = x_1 + x_2 - 1$ (line)
$\mathcal{T}f = \min\{\xi_1, \xi_2, 0\}$



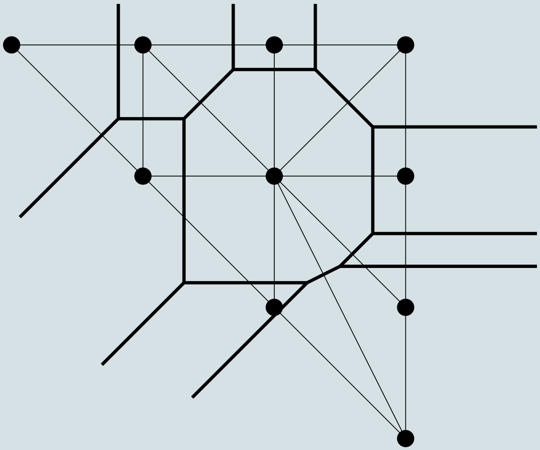
Plane Curves: conics





Plane Curves

A Cubic:



Mikhalkin (re)computed the number of *classical* degree d, genus g plane curves through $3d + g - 1$ general points:
count *tropical* such curves, each with a certain multiplicity.

Arbitrary codimension

I the ideal of $X \subseteq K^n$

Theorem 2 (EKL 2004, SS 2003, see also D 2006).

$$\mathcal{T}X = \{w \in \overline{\mathbb{R}}^n \mid \forall f \in I : \mathcal{T}f \textit{ not linear at } w\}$$

Theorem 3 (Bogart–Jensen–Speyer–Sturmfels–Thomas (2005)).
$\exists$ *finite subset of* I *for which previous theorem is true*

$\rightsquigarrow$ *tropical basis* (hard to compute!)
$\rightsquigarrow$ $\mathcal{T}X$ is a polyhedral complex

Theorem 4 (Bieri–Groves (1985), Sturmfels).
X *irreducible of dimension* $d \Rightarrow \dim \mathcal{T}X = d$

Example: $\mathcal{T}\mathrm{SL}_n$ and $\mathcal{T}\mathrm{O}_n$

Codimension one:
$\mathcal{T}\mathrm{SL}_n = \{A \in \overline{\mathbb{R}}^{n\times n} \mid \mathrm{tdet}\, A \leq 0$ and if < 0 then attained at least twice$\}$
monoid under tropical matrix multiplication

Higher codimension:
$\mathrm{O}_n := \{g \mid g^t g = I\}$
$\mathcal{T}\mathrm{O}_n = ??$
closed under tropical matrix multiplication?
not sufficient to tropicalise the n^2 defining equations
Observation (D-McAllister 2006):
$\mathcal{T}\mathrm{O}_n \supset \{$ n-point metrics $\}$ (full-dimensional cone)
tropical matrix multiplication = composition of metrics

Secant varieties

C a closed cone in a K-space V, $k \in \mathbb{N}$
$kC := \overline{\{v_1 + \ldots + v_k \mid v_i \in C\}}$,
the k-th *secant variety* of C
Main reference: Zak, *Tangents and secants of algebraic varieties*, 1993.

Example 5.

- $C_1 =$ rank ≤ 1 matrices in $V_1 = M_m$
  $kC_1 =$ rank $\leq k$ matrices
- $C_2 = \{z_1 \wedge z_2\} \subseteq V_2 := \bigwedge^2 K^m$
  cone over Grassmannian of 2-spaces in K^m
  $kC_2 =$ skew-symmetric matrices of rank $\leq 2k$
- $C_3 =$ cone over Grassmannian of *isotropic* 2-spaces in K^m
  $2C_3$ and $3C_3$ are complicated
  $kC_3 = kC_2$ for $k \geq 4$ (Baur and Draisma, 2004)

Non-defectiveness

Note: $\dim kC \leq \min\{k \dim C, \dim V\}$, the *expected dimension*.

Definition 6.
kC is *non-defective* if $\dim kC$ is as expected.
C is non-defective if all kC are.

Many Cs are non-defective, but hard to prove so!

Minimal orbits: interesting cones

V irrep of complex algebraic group G
$v \in V$ highest weight vector
$C := Gv \cup \{0\} \subseteq V$

all examples so far were of this form
dimensions of kC, $k = 1, 2, \ldots$ largely unknown!
(except $V = S^d(\mathbb{C}^n)$ for $G = \mathrm{GL}_n$—Alexander & Hirschowitz (1995))

widely open: tensor products, Grassmannians, etc.

Tropical strategy for proving kC non-defective

Recall :

algebraic geometry		tropical geometry
embedded affine variety $X \subseteq K^n$	$\to$	polyhedral complex $\mathcal{T}(X) \subseteq \overline{\mathbb{R}}^n$
polynomial map f	$\to$	piecewise linear map $\mathcal{T}(f)$
$\dim X$	$=$	$\dim \mathcal{T}(X)$

Strategy: prove $\dim \mathcal{T}(kC) = k \dim C$; then kC is non-defective.
But kC not known, let alone $\mathcal{T}(kC)$!
Proposal:

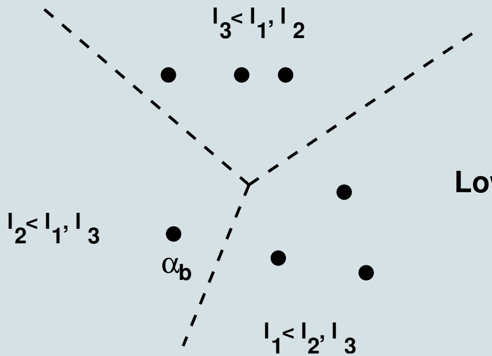
- parametrise $h : K^m \to C \subseteq V$
- tropicalise $f : (K^m)^k \to kC,\ (z_1, \ldots, z_k) \mapsto h(z_1) + \ldots + h(z_k)$
- compute $\operatorname{rk} dT(f)$ at a good point $\rightsquigarrow$ lower bound on $\dim \mathcal{T} kC$

A (simplified) theorem

$h = (h_1, \dots, h_n) : K^m \to C \subseteq K^n$ parametrisation
assume each $h_b = c_b x^{\alpha_b} \neq 0$ (1 term)

for $l = (l_1, \dots, l_k)$ k affine-linear functions on $\mathbb{R}^m$ set
$C_i(l) := \{\alpha_b | l_i(\alpha) < l_j(\alpha) \text{ for all } j \neq i\}$

Theorem 7 (Draisma, 2006). $\dim kC \geq \sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} C_i(l))$



Lower bound=3+2+1

Funny optimisation problem

$A \subseteq \mathbb{R}^n$ finite, $k \in \mathbb{N}$
Maximise $\sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} C_i(l))$=:*
over all $l = (l_1, \ldots, l_k)$, each l_i affine-linear

Corollary 8.
$A = \{\alpha_b \mid b\}$ *exponents of monomials in parametrisation*
draw A *on* m*-dimensional paper*
cut paper into k *pieces*
compute sg. like $*$
$\rightsquigarrow$ *lower bound on* $\dim kC$

Example from beginning:

C =Veronese cone
$h : (x_1, x_2, x_3) \mapsto (x_1 e_1 + x_2 e_2 + x_3 e_3)^d \in S^d(\mathbb{C}^3)$
satisfies conditions of theorem $\rightsquigarrow$ paper-and-scissors lower bound

Generalisation to higher dimensions?

Some results (with Karin Baur)

Non-degenerate:

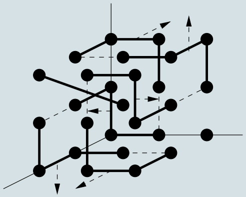
1. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except (even, 2)
2. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except (even, $1, 1$)
3. Segre embedding of $(\mathbb{P}^1)^6$ (cells for $k = 9$: 8 disjoint Hamming balls of radius 1 and one cell in the middle)

Almost done:

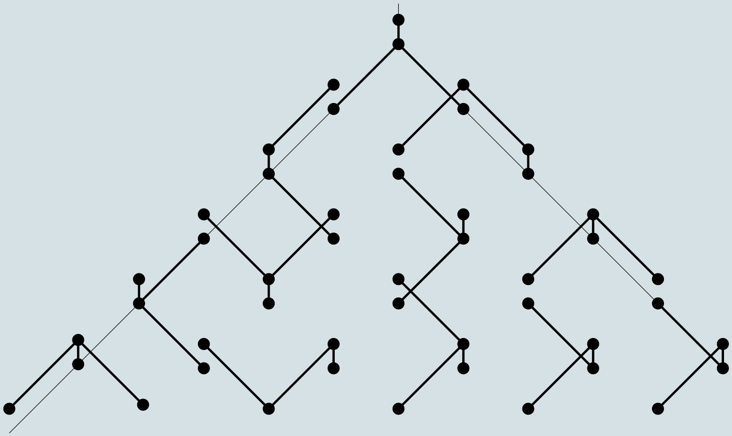
1. $\mathbb{P}^1 \times \mathbb{P}^2$
2. $\{$flags point $\subseteq$ line $\subseteq$ $\mathbb{P}^2\}$ probably all non-defective except those of highest weight $\omega_1 + \omega_2$ (adjoint representation) or $2\omega_1 + 2\omega_2$
   needs generalisation of Theorem 7

Conjecture: the lower bound always gives correct dimension for Segre-Veronese embeddings.

Some pictures



picture suggests: S-V embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2, 2, 2)$ has defective $7C$. Indeed!



Minimal orbit in representation of SL_3 of highest weight $(5, 1)$ is non-defective.

Conclusions

Non-defectiveness often provable by optimising a strange polyhedral-combinatoric objective function

Hope a point in $T(kC)$ with full-dimensional neighbourhood gives restrictions on the ideal of kC. Sufficient to settle one or two more cases of GSS?

Segre-Veronese is the given bound always correct?

Other minimal orbits Smallest flag variety has non-monomial parametrisation, but doable with a trick. In general: which parametrisation to use? (Littelmann-Bernstein-Zelevinsky polytopes?)

Tropical geometry is a powerful tool! (and interesting in its own right..)