A tropical approach to secant varieties

Jan Draisma

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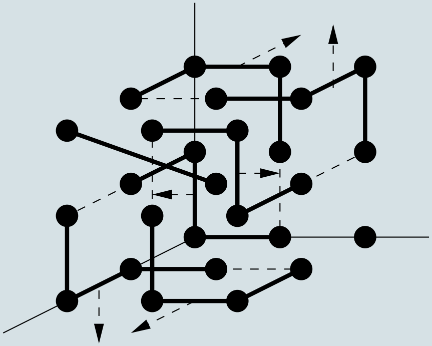
A vignette

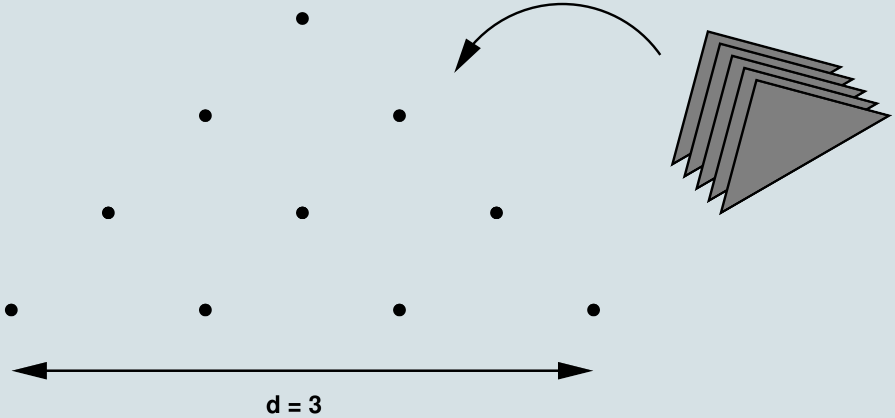


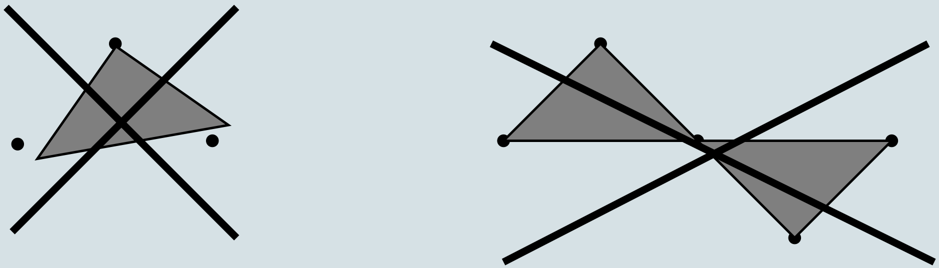
C = Cone over Segre(Veronese$_2(\mathbb{P}^1) \times$ Veronese$_2(\mathbb{P}^1) \times$ Veronese$_2(\mathbb{P}^1)$)

$kC = \overline{\{p_1 + \ldots + p_k \mid p_i \in C\}}$

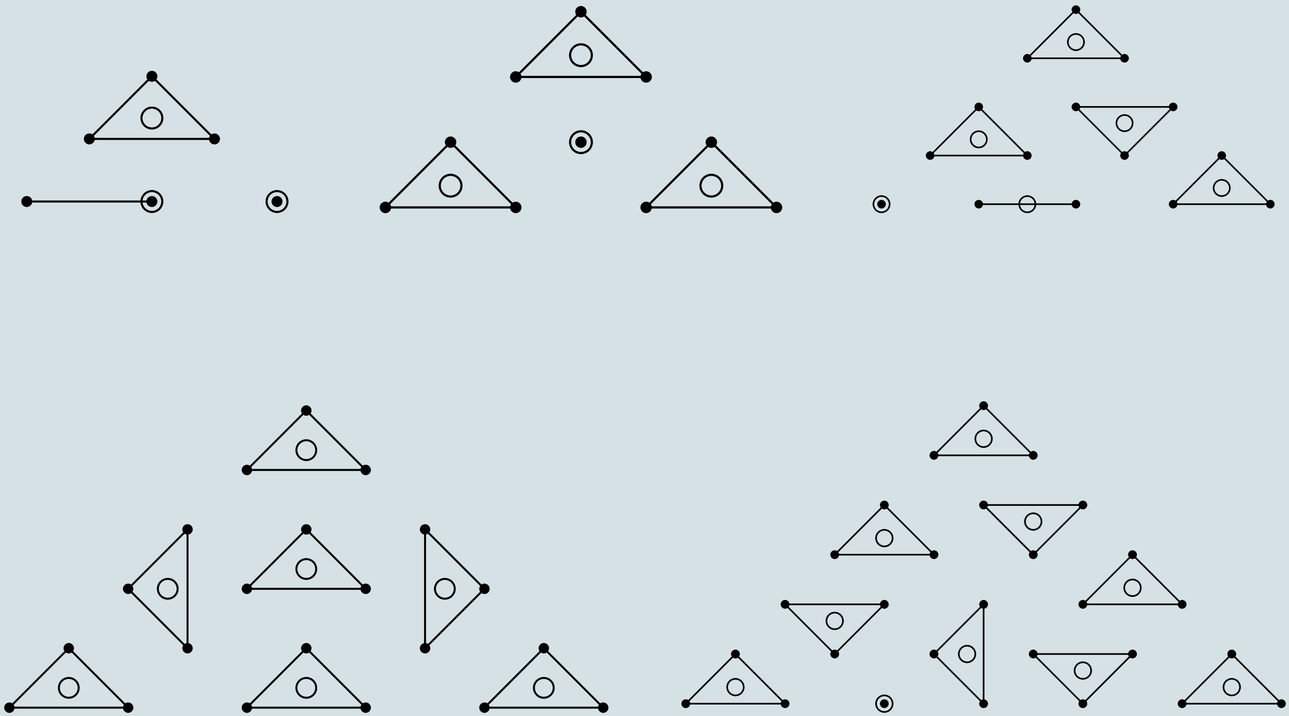
$\dim kC = (\geq)4, 8, 12, 16, 20, 24, 26, 27$

A jigsaw puzzle





Solutions for $d = 2, \ldots, 6$



A not so puzzling result

Theorem 1. d *is non-defective* unless

- $d = 2$ *(only* 1 *triangle fits), or*
- $d = 4$ *(only* 4 *triangles fit).*

Proof by induction.

Polynomial interpolation

Given: k generic points $p_1, \ldots, p_k \in \mathbb{P}^2$ and $d \in \mathbb{N}$.

Theorem 2 (Hirschowitz, 1985).
$\dim\{$ *homogeneous polynomials of degree* d *having singularities at all* p_i $\} =$ $\max\{0, \binom{d+2}{2} - 3k)\}$ unless

- $(d, k) = (2, 2)$: *dimension* = 1, *or*
- $(d, k) = (4, 5)$: *dimension* = 1.

Proof by induction with the *Horace method*.

A tautology?

Theorem 3. *Theorem 1 ⇒ Theorem 2*

Proof: *tropical geometry*!

Secant varieties

C a closed cone in a K-space V, $k \in \mathbb{N}$
$kC := \overline{\{v_1 + \ldots + v_k \mid v_i \in C\}}$,
the k-th *secant variety* of C
Main reference: Zak, *Tangents and secants of algebraic varieties*, 1993.

Example 4.

- $C_1 =$ rank ≤ 1 matrices in $V_1 = M_m$
  $kC_1 =$ rank $\leq k$ matrices
- $C_2 = \{z_1 \wedge z_2\} \subseteq V_2 := \bigwedge^2 K^m$
  cone over Grassmannian of 2-spaces in K^m
  $kC_2 =$ skew-symmetric matrices of rank $\leq 2k$
- $C_3 =$ cone over Grassmannian of *isotropic* 2-spaces in K^m
  $2C_3$ and $3C_3$ are complicated
  $kC_3 = kC_2$ for $k \geq 4$ (Baur and Draisma, 2004)

Non-defectiveness

Note: $\dim kC \leq \min\{k \dim C, \dim V\}$, the *expected dimension*.

Definition 5.
kC is *non-defective* if $\dim kC$ is as expected.
C is non-defective if all kC are.

Many Cs are non-defective, but hard to prove so!

Relation to polynomial interpolalation

$V = K^m$
$C := \{v^d\} \subseteq S^d V$
$\dim kC = ?$

Lemma 6 (Terracini, 1911). *For $v_1, \ldots, v_k \in V$ generic*
$\dim kC = \dim T_{v_1^d} C + \ldots + \dim T_{v_k^d} C.$

Lasker, 1904:
$T_{v_i^d} C = \{f \in S^d(V^*) \mid f \text{ is singular in } [v_i] \in \mathbb{P}V\}^0$
so
$\dim kC = \text{codim}\,\{\text{hom. pols. of degree } d \text{ singular in all } [v_i]\,\}.$

Alexander and Hirschowitz, 1995: $\dim kC$ for all k, d, m.
Secant dimensions of other classes of cones (e.g., Segre products and Grassmannians) still unknown!

Algebraic and tropical geometry

A rough guide:

algebraic geometry		tropical geometry
embedded affine variety $X \subseteq K^n$	$\longrightarrow$	polyhedral complex $T(X) \subseteq \overline{\mathbb{R}}^n$
polynomial map f	$\longrightarrow$	piecewise linear map $T(f)$
$\dim X$	$=$	$\dim T(X)$

Strategy: $\dim T(kC) = k \dim C \Rightarrow kC$ is non-defective.

But kC is not known! Solution:

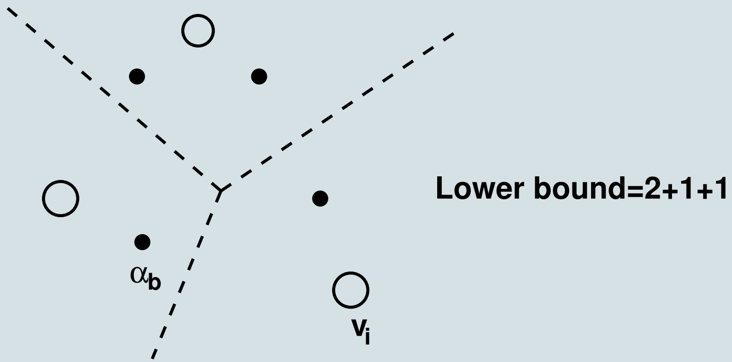
- parametrise $h : K^m \to C \subseteq V$
- tropicalise $f : (K^m)^k \to kC,\ (z_1, \ldots, z_k) \mapsto h(z_1) + \ldots + h(z_k)$
- compute $\operatorname{rk} dT(f)$ at a good point

A simplified theorem

$h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n$ parametrisation
Assume: each $h_b = c_b x^{\alpha_b} \neq 0$ (1 term).

Choose a 2-norm on $\mathbb{R}^m$.
For $v = (v_1, \ldots, v_k) \in (\mathbb{R}^m)^k$ set $\mathrm{Vor}_i(v) := \{\alpha_b$ in Voronoi cell of $v_i\}$.

Theorem 7 (Draisma, 2006). $\sum_i (1 + \dim \mathrm{Aff}_{\mathbb{R}} \mathrm{Vor}_i(v)) \leq \dim kC$.



Finally, a proof!

Proof of Theorem 3.
C =Veronese cone
$h : (x_1, x_2, x_3) \mapsto (x_1e_1 + x_2e_2 + x_3e_3)^d \in S^d(K^3)$
Combine theorem 7 and the jigsaw puzzle. □

Generalisation to higher dimensions?

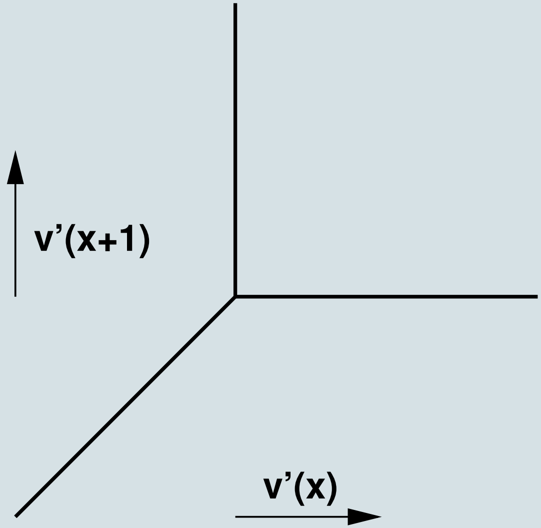
Tropical geometry

K a field with non-archimedean valuation $v : K \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$
X affine variety over K
$\bar{x} := (\bar{x}_1, \ldots, \bar{x}_n)$ generators of $K[X]$

Definition 8 (Tropicalisation of a variety).
$T_{\bar{x}}(X) := \{(v'(\bar{x}_1), \ldots, v'(\bar{x}_n)) \mid$
$v' : K[X] \to \overline{\mathbb{R}}$ valuation extending $v\}$

Depends on generators/embedding!

Example 9.
$X = \mathbb{A}^1,\ K[X] = K[x],$
$\bar{x}_1 = x,\ \bar{x}_2 = x + 1$



More concretely

L/K algebraically closed, complete extension with $v(L) = \overline{\mathbb{R}}$
I ideal of X in $K[x] = K[x_1, \ldots, x_n]$
For $w \in \overline{\mathbb{R}}^b$, $P = \sum_\alpha c_\alpha x^\alpha \in K[x]$:
$\mathrm{wt}_w P := \min_\alpha v(c_\alpha) + w \cdot \alpha$
$\mathrm{in}_w P :=$ terms of minimal weight

Theorem 10 (Einsiedler-Kapranov-Lind, Speyer-Sturmfels, see also D.). *The following are equal:*

- $v(X(L))$
- $T_{\bar{x}}(X)$
- $\{w \in \overline{\mathbb{R}}^n \mid \mathrm{in}_w f$ *is not monomial for any* $f \in I\}$

Theorem 11 (Bieri-Groves,Sturmfels). $T_{\bar{x}}(X)$ *is a polyhedral complex, pure of dimension* $\dim X$ *if* X *is irreducible.*

Definition 12 (Tropicalisation of polynomials).
$T(P)(w) := \mathrm{wt}_w P$
$T(h_1, \ldots, h_b) := (T(h_1), \ldots, T(h_b))$

Example 13.
$P = cx_1^2 + x_2$
$T(P) = \min\{c + 2w_1, w_2\}$

Lemma 14. *Given* $h : K^m \to C$
and $f(z_1, \ldots, z_k) := h(z_1) + \ldots + h(z_k)$*,*
$T(f)$ *maps* $(\mathbb{R}^m)^k$ *into* $T(kC)$.

Find a point where

- $T(f)$ is linear and
- $dT(f)$ has maximal rank.

This leads to Theorem 7.

Generalisation of Theorem 7

Optimisation problem:
Given
$k \in \mathbb{N}$
$A = (A_1, \ldots, A_n)$ list of finite subsets of $\mathbb{R}^n$
Optimisation domain
k-tuples $l = (l_1, \ldots, l_k)$ of affine linear functions on $\mathbb{R}^n$
Objective function
$\sum_{i=1}^{k}(1 + \dim \mathrm{Aff}_{\mathbb{R}}\, C_i(l))$
where $C_i = \bigcup_{b=1}^{n}\{\alpha \in A_b \mid f_i(\alpha) < f_j(\beta)$ for all $(\beta, j) \in A_b \times \{1, \ldots, k\}\}$
Optimal value $\mathrm{AP}^*(A, k)$

Theorem 15. $h = (h_1, \ldots, h_n) : K^m \to C \subseteq K^n$ *parametrisation*
$A_b \subseteq \mathbb{N}^m$ *support of* h_b
then $\mathrm{AP}^*(A, k) \leq \dim kC$

Some results (with Karin Baur)

Non-degenerate:

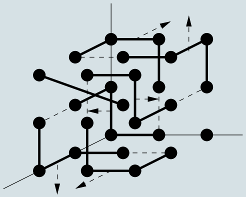
1. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 2)$
2. all Segre-Veronese embeddings of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ except $(\text{even}, 1, 1)$
3. Segre embedding of $(\mathbb{P}^1)^6$ (cells for $k = 9$: 8 disjoint Hamming balls of radius 1 and one cell in the middle)

Almost done:

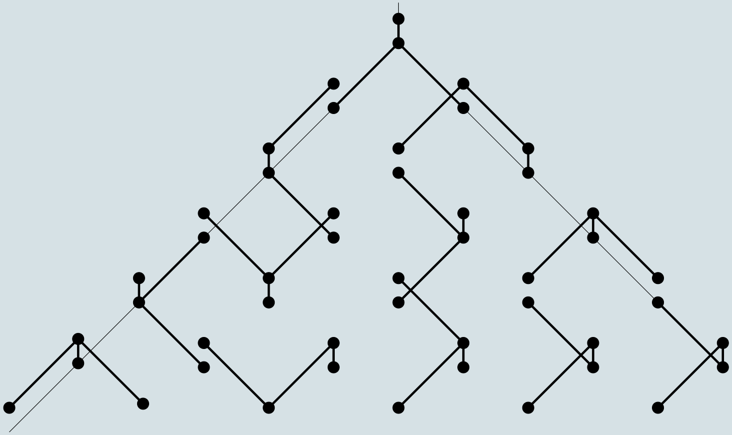
1. $\mathbb{P}^1 \times \mathbb{P}^2$
2. $\{$flags point $\subseteq$ line $\subseteq$ $\mathbb{P}^2\}$ probably all non-defective except those of highest weight $\omega_1 + \omega_2$ (adjoint representation) or $2\omega_1 + 2\omega_2$
   needs Theorem 15 rather than 7

Bold conjecture: the lower bound is always correct for Segre-Veronese embeddings.

Some pictures



picture suggests: S-V embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2,2,2)$ has defective $7C$. Indeed!



Minimal orbit in representation of SL_3 of highest weight $(5,1)$ is non-defective.

Conclusion

Non-defectiveness often provable by optimising a strange polyhedral-combinatoric objective function

Hope a point in $T(kC)$ with full-dimensional neighbourhood gives restrictions on the ideal of kC. Sufficient to settle one or two more cases of GSS?

Segre-Veronese is the given bound always correct?

Other minimal orbits Smallest flag variety doable with a trick: reduce all A_b in Theorem 15 to singletons, and use Voronoi-variant. In general: which parametrisation to use? (Littelmann-Bernstein-Zelevinsky polytopes?)

Reading

- Catalisano-Geramita-Gimigliano:
  some secant dimensions of Grassmannians, non-defectiveness of most $k(\mathbb{P}^1)^d$, defectiveness of some unbalanced Segre products and Segre-Veronese products.
- Sturmfels-Sullivant, Miranda-Dumitrescu, 200*:
  degeneration approach for secant dimensions
- Landsberg-Weyman, 2006:
  equations for certain secant varieties of Segre products (GSS conjectures!)
- Baur-Draisma-de Graaf, 2006:
  GAP-program for computing secant dimensions of *minimal orbits*
- Draisma, 2006:
  A tropical approach to secant dimensions `math.AG/0605345` (includes intro to tropical geometry!)

Hope this was not too non-linear