A scenic tour in tropical geometry

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Tropical Semiring

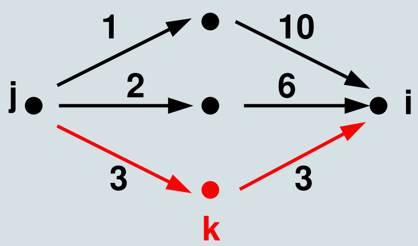
$\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$
$a \oplus b := \min\{a, b\}$
$a \odot b := a + b$

$\infty \oplus b = b$
$0 \odot b = b$
$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$

Getting used: tropical matrix multiplication

$A = (a_{ij})_{ij}, B = (b_{ij})_{ij} \in \overline{\mathbb{R}}^{n\times n}$
$(A \odot B)_{ij} := \min_k(a_{ik} + b_{kj})$

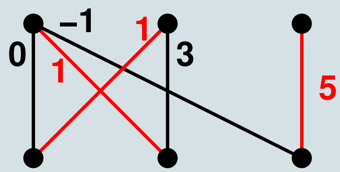
Application:
points $1, \ldots, n$
a_{ij} =distance from j to i
$\rightsquigarrow (A^{\odot k})_{ij}$ shortest length of a k–step path from j to i
if all $a_{ij} \geq 0$ and $a_{ii} = 0$ then $A^{\odot n}$ records all shortest path lengths
$\rightsquigarrow$ repeated squaring gives algorithm



Tropical polynomials

$I \subseteq \mathbb{N}^n$
$b_\alpha \in \overline{\mathbb{R}}$ for $\alpha \in I$
$f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}},$
$\xi \mapsto \bigoplus_{\alpha \in I} b_\alpha \odot \bigodot_{i=1}^n \xi_i^{\odot \alpha_i} = \min_{\alpha \in I} b_\alpha + \langle \xi, \alpha \rangle$ *tropical polynomial*

Example
$A \in \overline{\mathbb{R}}^{n \times n}$
$\mathrm{tdet}(A) := \bigoplus_{\pi \in S_n} a_{\pi(1),1} \odot a_{\pi(2),2} \odot \cdots \odot a_{\pi(n),n}$ *tropical determinant*
minimal weight matching in $K_{n,n}$ with edge weights a_{ij}



tdet =1+5+5

Tropical geometry

Set–up:
K field
$v : K \to \overline{\mathbb{R}}$ non–Archimedean valuation,
that is, $v^{-1}(\infty) = \{0\}$, $v(ab) = v(a) \odot v(b)$, and $v(a+b) \geq v(a) \oplus v(b)$
e.g. K =Laurent series and v=multiplicity of 0 as a zero
technical conditions on (K, v)

$X \subseteq K^n$ given by polynomial equations
$\rightsquigarrow \mathcal{T}X := \{v(x) = (v(x_1), \ldots, v(x_n)) \mid x \in X\}$
tropicalisation of X

depends on coordinates!

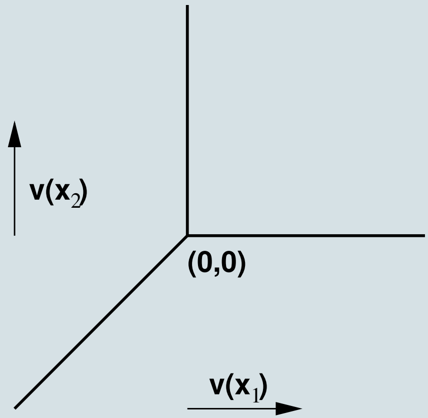
Codimension one varieties

X zero set of one polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$
$\mathcal{T} f(\xi) := \min_{\alpha \in \mathbb{N}^n}(v(c_\alpha) + \langle \xi, \alpha \rangle)$ tropicalisation of f

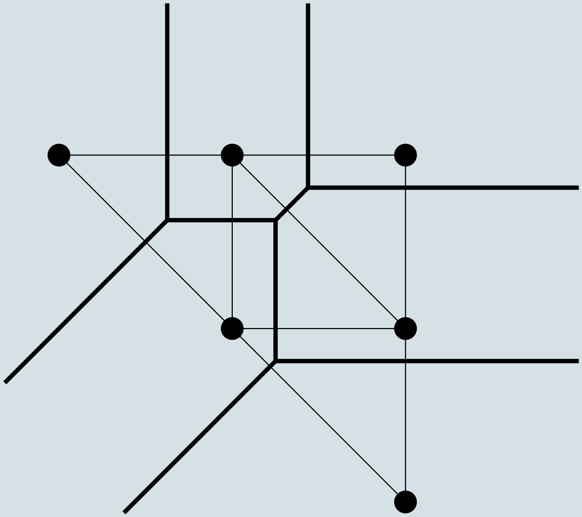
Theorem 1 (Einsiedler–Kapranov–Lind).
$\mathcal{T} X = \{\xi \in \overline{\mathbb{R}}^n \mid \mathcal{T} f \textit{ not linear at } \xi\}$

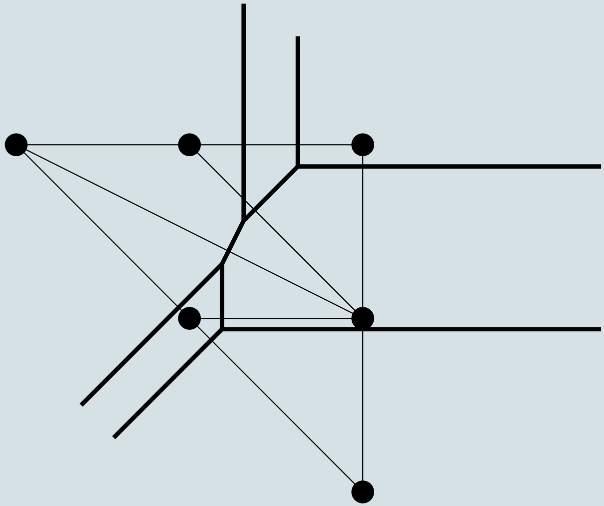
$\rightsquigarrow$ tropical hypersurfaces are polyhedral complexes!

Example:
$f = x_1 + x_2 - 1$ (line)
$\mathcal{T} f = \min\{\xi_1, \xi_2, 0\}$

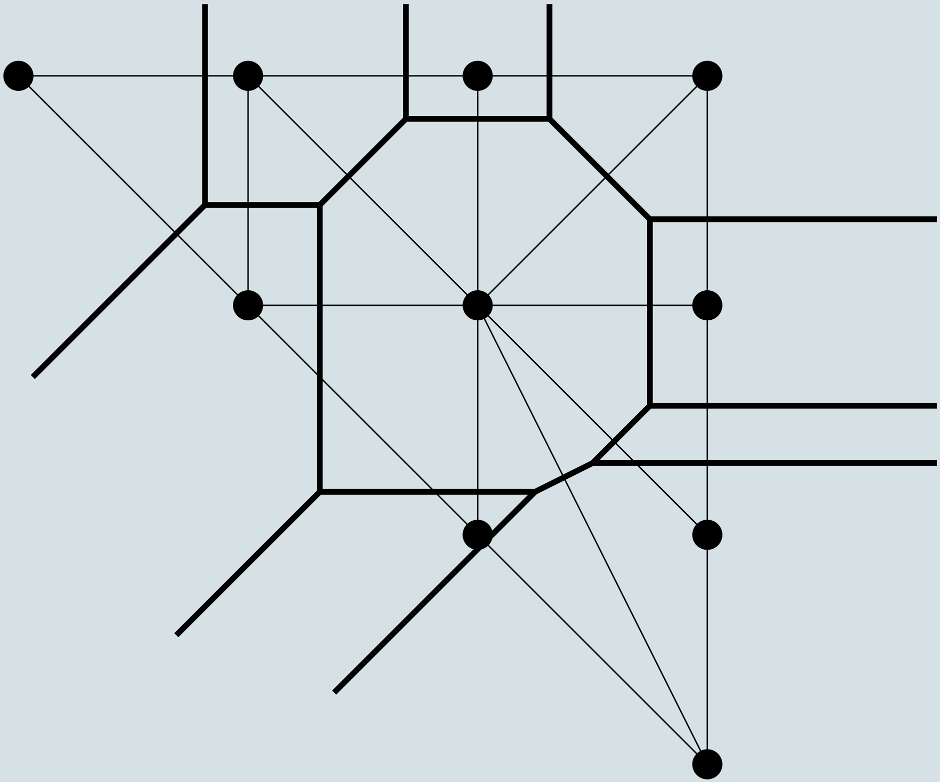


Plane Curves: conics





Plane Curves: a cubic



Plane curves: counting

Proposition 2. *characterisation of tropical curves of degree d in the plane:*

1. *plane graph, straight edges with multiplicities and rational slopes*
2. *balancing condition at vertices*
3. *d infinite tentacles in each direction* $(-1, 0)$, $(0, 1)$, $(1, 1)$

Mikhalkin (re)computed the number of *classical* degree d, genus g plane curves through $3d + g - 1$ general points:
count *tropical* such curves, each with a certain multiplicity.

Caporaso–Harris in 1998 needed havier algebraic geometry!

algorithms for enumerating such tropical curves
(Mikhalkin, Gathmann–Markwig)

Arbitrary codimension

I the ideal of $X \subseteq K^n$

Theorem 3 (EKL 2004, SS 2003, see also D 2006).

$$\mathcal{T}X = \{w \in \overline{\mathbb{R}}^n \mid \forall f \in I : \mathcal{T}f \textit{ not linear at } w\}$$

Theorem 4 (Bogart–Jensen–Speyer–Sturmfels–Thomas (2005)).
$\exists$ *finite subset of* I *for which previous theorem is true*

$\rightsquigarrow$ *tropical basis* (hard to compute!)
$\rightsquigarrow$ $\mathcal{T}X$ is a polyhedral complex

Theorem 5 (Bieri–Groves (1985), Sturmfels).
X *irreducible of dimension* $d \Rightarrow \dim \mathcal{T}X = d$

Application: polynomial interpolation in two variables

Set–up:
$d \in \mathbb{N}$
$p_1, \dots, p_k$ general points in $\mathbb{C}^2$
$\operatorname{codim}\{f \in \mathbb{C}[x,y]_{\leq d} \mid \forall i : f(p_i) = f_x(p_i) = f_y(p_i) = 0\} = ??$
expect: $\min\{3k, \binom{d+2}{2}\}$ (upper bound)

Hirschowitz (1985):
correct, unless $(d,k) = (2,2)$ or $(d,k) = (4,5)$ (1 instead of 0)

D (2006): new proof using tropical geometry, paper and scissors

Alexander and Hirschowitz: more variables (1995)
Also doable tropically??

Some progress

1. Tropical Grassmannian of lines $\rightsquigarrow$ space of phylogenetic trees (Speyer and Sturmfels, 2003)
2. Tropical geometry of statistical models (Pachter, Sturmfels, 2004)
3. Tropical Pappus theorem (Tabera, 2003)
4. Tropical discriminants (Dickenstein, Feichtner, Sturmfels, 2005)
5. Tropical Bézout theorem (Richter–G./Sturmfels/Theobald, Gathmann)
6. Tropical relative Gromov–Witten invariants (Gathmann, Markwig)
7. The cone of n–point metrics is a cone in the tropical orthogonal group (D (2006))
8. Secant dimensions of low–dimensional homogeneous varieties in high–dimensional projective spaces (Baur–D, ongoing)
9. Polyhedral–combinatorial (paper and scissors) programs related to these dimensions (D–Halupczok, ongoing)

Lots of work left to be done!

Theory:

1. better proof for existence of tropical bases
2. gluing tropical variaties?
3. tropical morphisms?
4. relation to Berkovich theory

Applications/computations:

1. algorithms for computation of tropical bases
2. algorithms for enumerative tropical geometry
3. tropicalisations of algebraic groups
4. further applications to algebraic statistics and mathematical biology