# Tropical Algebraic Groups ?!

Jan Draisma (j.w.i.p.w. Tyrrell McAllister)

Göttingen, 9 July 2008

# Ph.D. position in Eindhoven/Bremen

`http://www.win.tue.nl/~jdraisma/index.php?location=tag`
or Google "Tropical Algebraic Groups" and find me.

# Algebraic Groups

**Definition.** *algebraic group $G$:* variety + compatible group structure.

**Lemma.** *$G$ affine $\Rightarrow$ closed subgroup of some $\mathrm{GL}_n$.*

This project: only affine groups
(as opposed to Abelian varieties).

**Example.**
$\mathrm{SL}_n = \{g \in M_n \mid \det(g) = 1\}$
$\mathrm{O}_n = \{g \in M_n \mid g^T g = I\}$
$\mathrm{U}_n = \{g \in M_n \mid g_{ii} = 1, g_{ij} = 0 \text{ for all } i > j\}$

Algebraic groups describe symmetries of beautiful varieties such as Segre-Veronese embeddings of projective spaces and Grassmannians.

# Goal

**Goal of the project:** tropicalise affine algebraic groups and their actions on varieties—but how? Tropicalisation depends on coordinates!

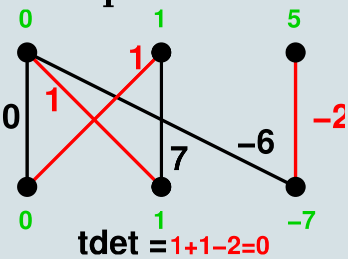
**Remark.**
*Berkovich space* $\mathcal{B}G$ of $G$ = "all possible tropicalisations of $G$" doesn't. But $\mathcal{B}(G \times G) \neq \mathcal{B}(G) \times \mathcal{B}(G)$, so Berkovichisation of multiplication is not a map $\mathcal{B}(G) \times \mathcal{B}(G) \to \mathcal{B}(G)$.

**Provisional remedy:** work in nice coordinates.

# A nice example

$\mathcal{T}\mathrm{SL}_n = \{w \in M_n(\overline{\mathbb{R}}) \mid$ minimal weight of a perfect matching is $\leq 0$ and attained twice if $< 0$ $\}$

**Example.**

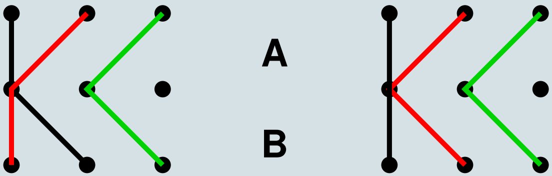



**Theorem (Egerváry, 1931).** $\exists a_1, \ldots, a_n, b_1, \ldots, b_n : w_{ij} \geq a_i + b_j$ *with* $=$ *on the edges of some* $w$*- minimal perfect matching.*

$\rightsquigarrow$ description of full- dimensional cones of $\mathcal{T}\mathrm{SL}_n$

**Proposition.** $\mathcal{T}\mathrm{SL}_n$ *is a monoid under tropical matrix multiplication.*

*Proof.*




□

# An example that ought to be nice

$O_n = \{g \in M_n \mid g^T g = I\}$; what is $\mathcal{T}O_n$??

**Challenges:**

- a tropical basis?
  (equations above, $\det(g) = \pm 1$, $g_{ij} = det(g_{[n]-i,[n]-j})$?)
- is it a monoid?

**Definition.** $D = (d_{ij})$ is *metric* if $d_{ii} = 0, d_{ij} = d_{ji}, d_{ij} + d_{jk} \geq d_{ik}$.

**Proposition.** *$\mathcal{T}O_n$ contains the cone of all metric matrices.*

*Proof.* Use the *Lie algebra* of $O_n$. Consider

$$\exp \begin{bmatrix} 0 & x_{12} & \dots & x_{1n} \\ -x_{12} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & x_{n-1,n} \\ -x_{1n} & \dots & -x_{n-1,n} & 0 \end{bmatrix} \text{ where } x_{ij} \text{ has valuation } d_{ij}$$

□

# Total positivity

**Example.** $\mathrm{GL}_n^+(\mathbb{R}_{\geq 0}) :=$ monoid in $\mathrm{GL}_n(\mathbb{R})$ generated by

- diagonal matrices with positive diagonal entries, and
- the $1$- parameter groups $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$:

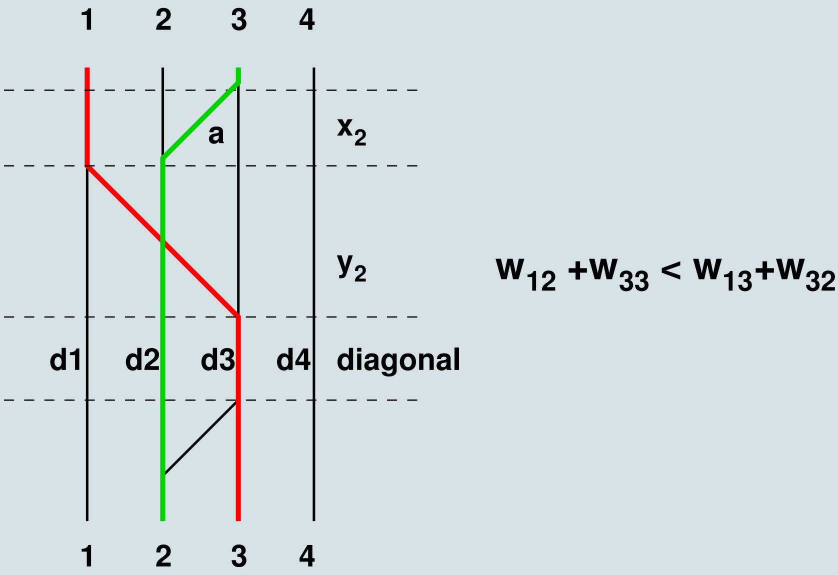
$$x_1(t) = \begin{bmatrix} 1 & t & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \text{ with } t > 0, y_1 = x_1^T, \text{ etc.}$$

**Theorem (classical).**
$\mathrm{GL}_n^+(\mathbb{R}_{\geq 0}) = \{g \in \mathrm{GL}_n(\mathbb{R}) \mid$ *all determinants of square submatrices of $g$ are* $\geq 0\}$.

**Example (continued).** $\mathcal{T}^{+}\mathrm{GL}_n :=$ submonoid of $M_n(\overline{\mathbb{R}})$ generated by

- diagonal matrices without $\infty$ on the diagonal
- the $x_i(a)$, $y_i(a)$ with $a \in \overline{\mathbb{R}}$.

**Proposition.**
$\mathcal{T}^{+}\mathrm{GL}_n = \{w \in M_n(\overline{\mathbb{R}}) \mid w_{ij} + w_{kl} \leq w_{il} + w_{kj} \textit{ if } i < k, j < l\}$,




*Proof of* $\subseteq$. □

# Positive monoid

$F$: a zero- sum- free semifield
$\Gamma$: a Dynkin diagram

**Definition.** $G^+(F) :=$ monoid presented by

- generators: $x_i(a), t_i(b), y_i(c)$ for vertices $i$ of $\Gamma$, $a, c \in F$, $b \in F^*$.
- relations mimicking those in $\mathrm{GL}_n^+$, e.g.

$$x_i(0) = 1 \quad x_i(a)x_i(b) = x_i(a+b)$$
$$t_i(ab) = t_i(a)t_i(b) \quad t_i(1) = 1$$
$$x_i(a)x_j(b) = x_j(b)x_i(a) \text{ for } i \not\sim j$$
$$x_i(a)x_j(b)x_i(c) = x_j(bc/(a+c))x_i(a+c)x_j(ab/(a+c)) \text{ for } i \sim j$$
$$y_i(b)x_i(a) = x_i(a/(ab+1))t_i(1/(ab+1))y_i(b/(ab+1))$$

*Lusztig variety* (Berenstein- Fomin- Zelevinsky).

# Positive Monoid

**Remark.** $G^+$ is a functor from zero- sum- free semifields to monoids.

**Example.**

$$y_i(b)x_i(a) = x_i(a/(ab+1))t_i(1/(ab+1))y_i(b/(ab+1)) \text{ comes from}$$
$$\begin{bmatrix}1 & 0\\ b & 1\end{bmatrix}\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix} = \begin{bmatrix}1 & \frac{a}{ab+1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{ab+1} & 0\\ 0 & ab+1\end{bmatrix}\begin{bmatrix}1 & 0\\ \frac{b}{ab+1} & 1\end{bmatrix}$$

**Example.**
$\Gamma$ =path of length $n-1$ and $F = \mathbb{R}_{\geq 0}$ gives $\mathrm{GL}_n^+$,
but for $F = \overline{\mathbb{R}}$ the map $G(\overline{\mathbb{R}})^+ \longrightarrow M_n(\overline{\mathbb{R}})$ is not injective.

**Proposition (Lusztig).**
*$\exists$ finite word $x_{i_1} \cdots x_{i_l} t_1 \cdots t_n y_{i_1} \cdots y_{i_l}$ such that $F^l \times (F^*)^n \times F^l \longrightarrow G^+(F)$ surjective*

# Representations of the positive monoid

$G$: simply connected group over $\mathbb{C}$ with diagram $\Gamma$
Every $G$- representation $V$ of $G$ has a *canonical basis*.
In these coordinates the generators $x_i, t_i, y_i$ make sense over any zero- sum-free semifield $F$.
**Observation:** image of $G^+(\overline{\mathbb{R}}) \to M_n(\overline{\mathbb{R}})$ is contained in $\mathcal{T}G$ (take $F =$ Puiseux series with positive leading coefficient and use functoriality).
**Sample Questions:**

- Is Speyer- Williams' totally positive Grassmannian one orbit under $\mathrm{GL}_n^+(\overline{\mathbb{R}})$?)
- $\exists$ representation $V$ faithful for all $F$? (I think yes; product of fundamental representations?)
- Describe the polyhedral complexes that are the images of $G^+(\overline{\mathbb{R}})$ in representations.

$\rightsquigarrow$ these should be our "nice coordinates"

# A “naive” approach

**Definition.** An additive tropical 1- PSM is a tropical map $x : \overline{\mathbb{R}} \to M_n(\overline{\mathbb{R}})$ such that $x(\infty) = I$ and $x(a+b) = x(a)x(b)$.

**Lemma.** *$x$ is upper triangular up to a permutation.*

**Questions:**

- Given finitely many 1- PSM’s $x_i$, what monoid do they generate?
- $\exists$ finite word in the $x_i$ surjective on the monoid?
- Is the monoid pure?

**Proposition.** *All $x_i$ simultaneously upper- triangular $\Rightarrow \exists$ a finite word surjective on the monoid.*