Technische Universiteit
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Tensors of bounded rank

Jan Draisma

12 April 2012
Nederlands Mathematisch Congres

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What's a tensor?

or

$$\sum_{i=1}^{k} v_{i1} \otimes \cdots \otimes v_{ip} \in V_1 \otimes \cdots \otimes V_p$$

Matrix rank

Definition

rank of $\omega \in V_1 \otimes V_2$ is the minimal k in $\omega = \sum_{i=1}^{k} v_{i1} \otimes v_{i2}$

Facts

- equals matrix rank
- relevant for applications (SVD)
- efficiently computable
- $\operatorname{rk} \omega \leq k \Leftrightarrow (k+1) \times (k+1)$ subdeterminants of ω are zero

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- relevant for applications (SIAM activity group in AG!)
- NP-hard (Håstad, Hillar-Lim)

Example (Strassen)
2×2 matrix multiplication $\in \mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ has rank 7 instead of 8

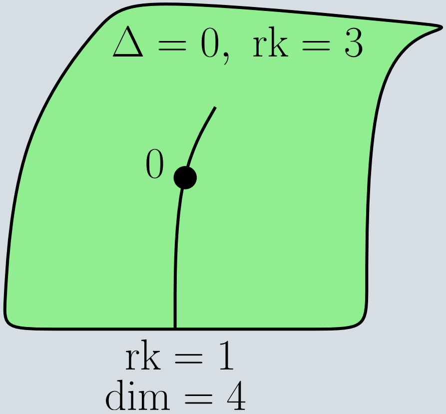
$n \times n$ matrix multiplication:
$O(n^{\log_2 7})$ scalar multiplications

Not closed

Question
characterisation of $\{\mathrm{rk} \leq k\}$
by equations?

Example
$\omega = e_1 \otimes A + e_2 \otimes B \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
$\Delta =$ discriminant $(\det(xA + yB))$

$\Delta \neq 0, \ \mathrm{rk} = 2$



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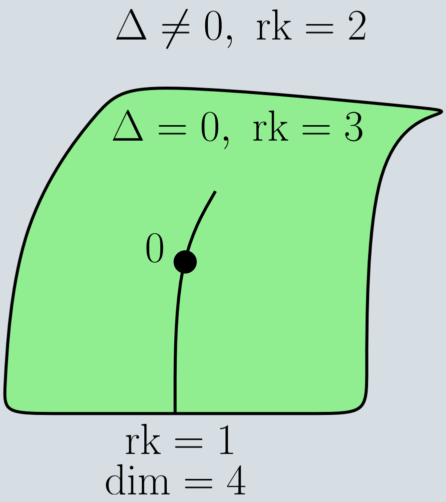


Border rank

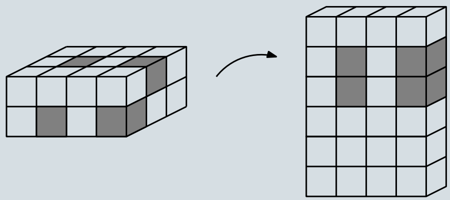
Definition
ω has *border rank* $\le k$
if $\omega \in \overline{\{\mathrm{rk} \le k\}}$

so $\mathrm{brk}(\omega) \le 2$ for $\omega \in (\mathbb{C}^2)^{\otimes 3}$

Example (Landsberg)
2×2 matrix multiplication
has $\mathrm{brk} = 7$

Flattening

$V_1 \otimes \cdots \otimes V_p \to$
$(V_1 \otimes \cdots \otimes V_q) \otimes (V_{q+1} \otimes \cdots \otimes V_p)$
does not increase rank



$(k+1) \times (k+1)$-determinants
of flattenings vanish on $\{\mathrm{brk} \leq k\}$

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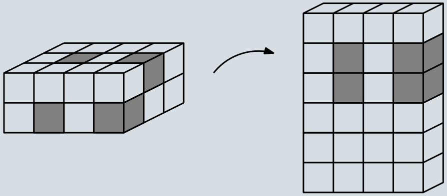

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Bounded degree

Theorem (D-Kuttler)
For *fixed* k:
- there exists $d = d(k)$ such that for all $p, V_1, \ldots, V_p$ the set $\{\mathrm{brk} \leq k\}$ is defined by the vanishing of polynomials of degree at most $d(k)$.
- $\mathrm{brk} \leq k$ can be tested in polynomial time.

Known

k	$d(k)$	
0	1	
1	2	Segre(?)
2	3	Landsberg-Manivel/Raicu
4	≥ 9	Friedland-Gross/ Bates-Oeding

Strassen's hypersurface

(following Ottaviani)

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$$

$$\Big\downarrow (x,y,z) \mapsto \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}$$

$$\mathbb{C}^3 \otimes (\mathbb{C}^3 \otimes \mathbb{C}^3) \otimes \mathbb{C}^3$$

$$\Big\downarrow$$

$$(\mathbb{C}^3 \otimes \mathbb{C}^3) \otimes (\mathbb{C}^3 \otimes \mathbb{C}^3)$$

$$\Big\downarrow$$

$$\mathbb{C}^9 \otimes \mathbb{C}^9$$

maps rank 1 to rank 2
hence rank ≤ 4 to rank ≤ 8
take determinant!

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maps rank 1 to rank 2
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Bounded degree, proof sketch

- w.l.o.g. all $V_i = V := \mathbb{C}^{k+1}$
- $X_p := \{\mathrm{brk} \leq k\} \subseteq V^{\otimes p}$
- $Y_p \subseteq V^{\otimes p}$ defined by $(k+1) \times (k+1)$-determinants of flattenings

$\rightsquigarrow X_p \subseteq Y_p \subseteq V^{\otimes p}$

- construct $T_\infty = \lim_{\leftarrow p} V^{\otimes p}$
- contains $Y_\infty = \lim_{\leftarrow p} Y_p$
- contains $X_\infty = \lim_{\leftarrow p} X_p$

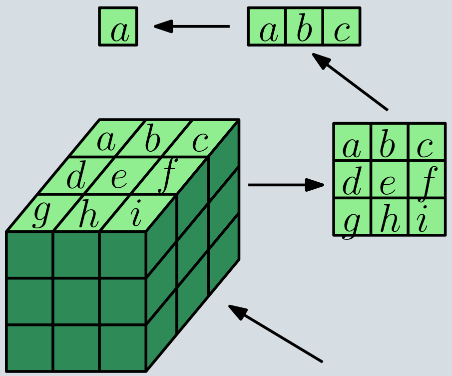
$\rightsquigarrow X_\infty \subseteq Y_\infty \subseteq T_\infty$

Tensors: the limit

fix $x_0 \in V^*$
$V^{\otimes p+1} \to V^{\otimes p}$,
$v_1 \otimes \cdots \otimes v_{p+1}$
$\mapsto x_0(v_{p+1}) v_1 \otimes \cdots \otimes v_p$

maps $X_{p+1} \to X_p$
and $Y_{p+1} \to Y_p$

An element of T_∞



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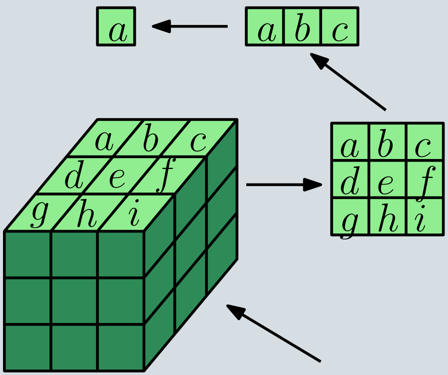

Bounded degree, proof sketch

Highly symmetric
on $T_\infty, X_\infty, Y_\infty$ acts
$G_\infty := \bigcup_p (\mathrm{Sym}(p) \ltimes \mathbf{GL}(V)^p)$

Last step
Show that $Y_\infty(\mathbb{C})/G_\infty$ is
Noetherian. □