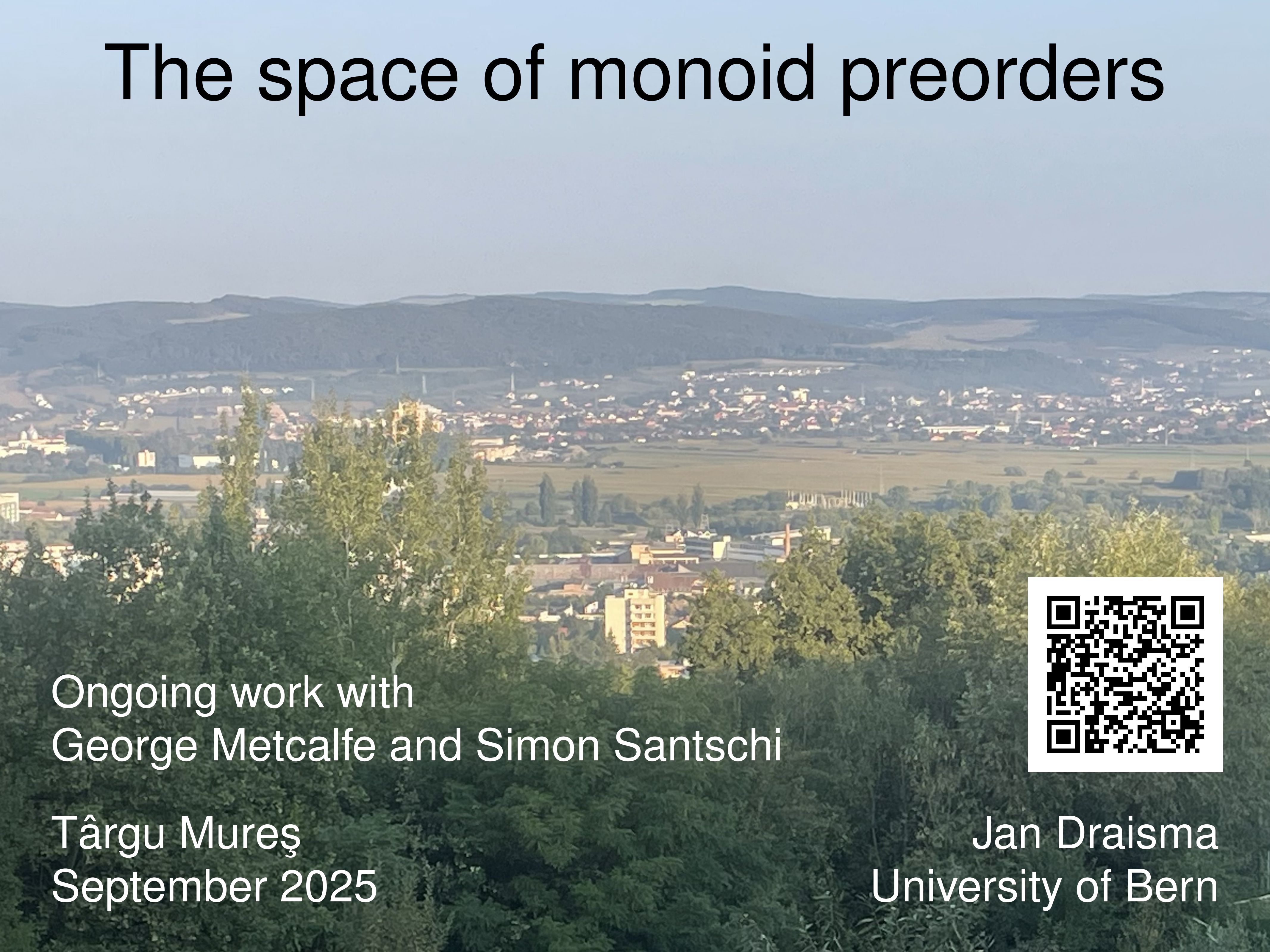

The space of monoid preorders

Ongoing work with
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A question from universal algebra

Question [Metcalfe]

$\exists$? an algorithm that solves systems such as

$$\max\{x_1x_2x_3, x_4x_5x_6, x_7x_8x_9\} < \min\{x_1x_4x_7, x_2x_5x_8, x_3x_6x_9\}$$

... for the unknown *preorder* $\leq$ on $\{x^a \mid a \in \mathbb{Z}^9_{\geq 0}\}$?

Setting

$(\Pi, \cdot)$: commutative monoid with neutral element 1

Definition

A *preorder* $\leq$ on Π:

- $(u \leq v \text{ and } v \leq w) \Rightarrow u \leq w$
- $u \leq u$
- $u \leq v$ or $v \leq u$
- $u \leq v \Rightarrow uw \leq vw$

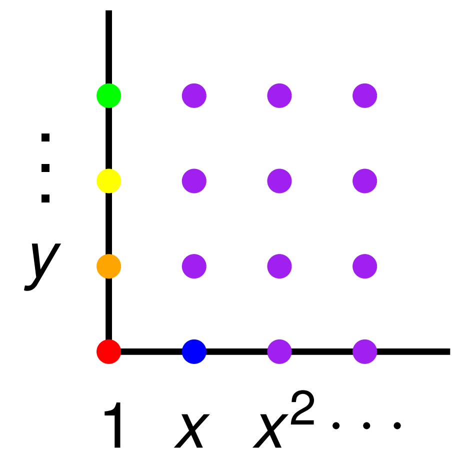
we do *not* require:

- $(u \leq v \text{ and } v \leq u) \Rightarrow u = v$
- $1 \leq u$

- $(\Pi, \cdot) = (\mathbb{R}^k, +)$ and $u \leq_{\text{lex}} v :\Leftrightarrow u = v$ or the first nonzero entry of $u - v$ is < 0.
- $\rho : \Pi \to \Pi'$ a homomorphism, $\leq'$ a preorder on Π' $\rightsquigarrow$ $\leq := \rho^*(\leq')$ by $u \leq v :\Leftrightarrow \rho(u) \leq' \rho(v)$ is a preorder on Π.

Theorem [Robbiano, 1985]
For any preorder $\leq$ on $(\mathbb{Z}^n, +)$ there exist $k \leq n$ and a homomorphism $\rho : \mathbb{Z}^n \to \mathbb{R}^k$ such that $\leq = \rho^*(\leq_{\text{lex}})$.

- $\text{Mon}_2 = \{x^i y^j \mid i, j \geq 0\} \cong \mathbb{Z}^2_{\geq 0}$
$\leq$ means further up the rainbow here:



Question
Characterisation of preorders on Mon_n *à la* Robbiano?

The role of monomial and binomial ideals

Definition: An *ideal* in Π is a subset I with $\Pi \cdot I \subseteq \Pi$.

• For $\Pi = \mathrm{Mon}_n \rightsquigarrow$ *monomial ideals* in $K\Pi = K[x_1, \dots, x_n]$.

• If $1 \le$ all $u \in \Pi$, then each up-set is an ideal; conversely, can construct positive preorders from certain ideals.

Important observation

$\le$ a preorder on $\Pi \rightsquigarrow I_\le := \langle \{u - v \mid u \le v \le u\} \rangle_K$ is a *binomial* ideal in $K\Pi$ (in particular finitely generated).

• If $I_\le$ is *prime*, then $\le$ is the pull-back of a preorder on the *groupification* $\mathrm{Gr}(\Pi)$ ($\cong \mathbb{Z}^n$ when $\Pi = \mathrm{Mon}_n$).

• In general: $\le$ is described by $I_\le$ and an *order* on monomials in $K\Pi / I_\le$. Surely *orders* on a monoid Π are easy to describe?

The space of preorders

Notation

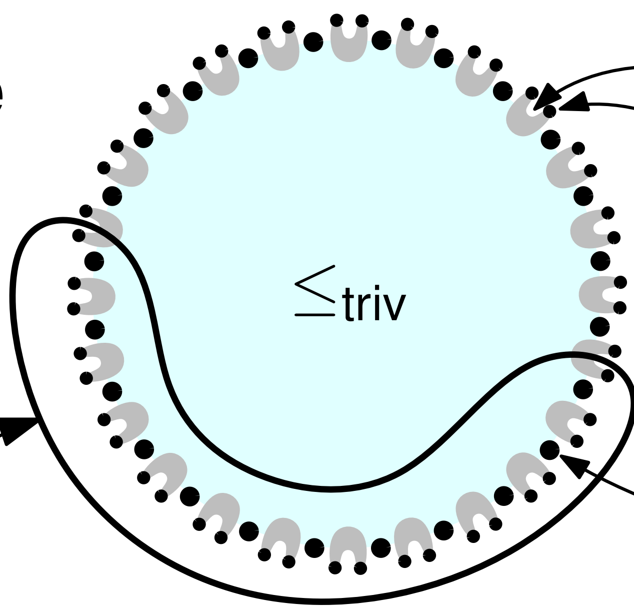
- $u \approx v$ means ($u \leq v$ and $v \leq u$)
- $u < v$ means ($u \leq v$ and not $v \leq u$)

Definition

Π a commutative monoid $\rightsquigarrow \mathcal{P}(\Pi) = \{$all preorders on $\Pi\}$ + weakest topology with $\{\leq \mid u < v\}$ *closed* for all u, v.

Example

$\Pi = \mathbb{Z}^2$:



$(u_1, u_2) \leq (v_1, v_2) \Leftrightarrow$

$u_1 + u_2 \leq v_1 + v_2$

$\rho u^T \leq_{\text{lex}} \rho v^T$ with $\rho = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\sqrt{3} \cdot u_1 - u_2 \leq \sqrt{3} \cdot v_1 - v_2$

$\{\leq \mid (1, 2) < (0, 0)\}$

Spectrality

Theorem A

• For any finitely generated commutative monoid Π, $\mathcal{P}(\Pi)$ is spectral, and every point is open in its closure.

• For all $n \geq 1$, $\mathcal{P}(\mathbb{Z}^n)$ has Krull dimension n but $\mathcal{P}(\mathbb{Z}^n_{\geq 0})$ has infinite Krull dimension.

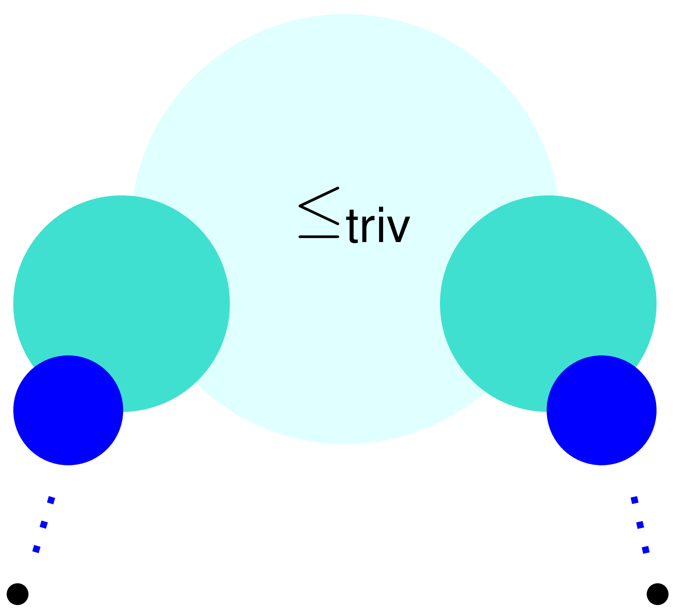
Example

$\mathcal{P}(\mathbb{Z}_{\geq 0})$:

$0 > 1 \approx 2 \approx 3 \approx \cdots$

$0 > 1 > 2 \approx 3 \approx \cdots$

$0 > 1 > 2 > 3 > \cdots$



$0 < 1 \approx 2 \approx 3 \approx \cdots$

$0 < 1 < 2 \approx 3 \approx \cdots$

$0 < 1 < 2 < 3 < \cdots$

Main results

Definition
A $\mathbb{Q}$-*constructible set* in $\mathbb{R}^n$ is a set of the form $A = S \setminus H$ with S semi-algebraic over $\mathbb{Q}$ and H a countable union of hyperplanes through 0 defined over $\mathbb{Q}$.

Theorem B
For any finitely generated commutative monoid Π, there exist $\mathbb{Q}$-constructible sets $A_1 \subseteq \mathbb{R}^{n_1}, A_2 \subseteq \mathbb{R}^{n_2}$, ... and continuous maps $\varphi_i : A_i \to \mathcal{P}(\Pi)$ such that $\mathcal{P}(\Pi) = \bigcup_i \mathrm{im}(\varphi_i)$.

Theorem C
There exists an algorithm that on input Π and a finite boolean combination of basic closed sets $\{\leq \mid u < v\} \subseteq \mathcal{P}(\Pi)$ decides whether that combination is empty.

(Necessary) generalisation

Definition

- A Π-*set* S is a set with a map $\Pi \times S \to S$, $(u, s) \mapsto us$ s.t. $1s = s$ and $(uv)s = u(vs)$; S is *fin. gen.* if $S = \bigcup_{i=1}^{k} \Pi s_i$.
- A *preorder* on S: $s \leq t \Rightarrow us \leq ut$. (no preorder on Π)

Theorems A,B,C hold for the space $\mathcal{P}(S)$ of preorders on S, for any f.g. commutative monoid Π and any f.g. Π-set S.

Note: $\Pi = \mathrm{Mon}_n$, $S = \mathrm{Mon}_n \times \{1, \ldots, m\} \rightsquigarrow \mathcal{P}(S) = \{$monomial preorders for the free module $K[x_1, \ldots, x_n]^m\}$.

Rest of this talk:
1. the case where Π is a group (generalisation of Robbiano's Theorem to Π-sets); and 2. the monoid case.

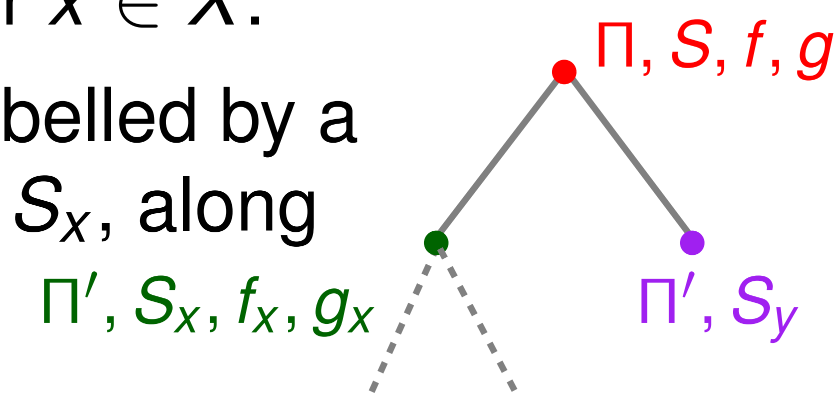
Setting: Π a f.g. abelian group and S a f.g. Π-set.

Proposition

$\leq$ on S nontrivial $\rightsquigarrow$ $\exists f : \Pi \to \mathbb{R}$ homo, $g : S \to \mathbb{R}$ nonconstant s.t. $g(us) = f(u) + g(s)$ and $s \leq t \Rightarrow g(s) \leq g(t)$.

Induction: $\Pi' := \ker(f)$, $X \subseteq \mathbb{R}$ finite set of $\mathrm{im}(f)$-coset representatives in $\mathrm{im}(g)$, $S_x := g^{-1}(x)$ $\rightsquigarrow$ $\leq$ is determined by g and $\leq|_{S_x}$ on the Π'-sets S_x for $x \in X$.

$\rightsquigarrow$ Get a *preotree* for $\leq$: vertex x labelled by a subgroup $\Pi_x \subseteq \Pi$ and a f.g. Π_x-set S_x, along with *numerical data* (f_x, g_x).



Theorem [DMK, Rust-Reid 97,...]**:** Every preorder on S comes from a preotree with numerical data, and vice versa.

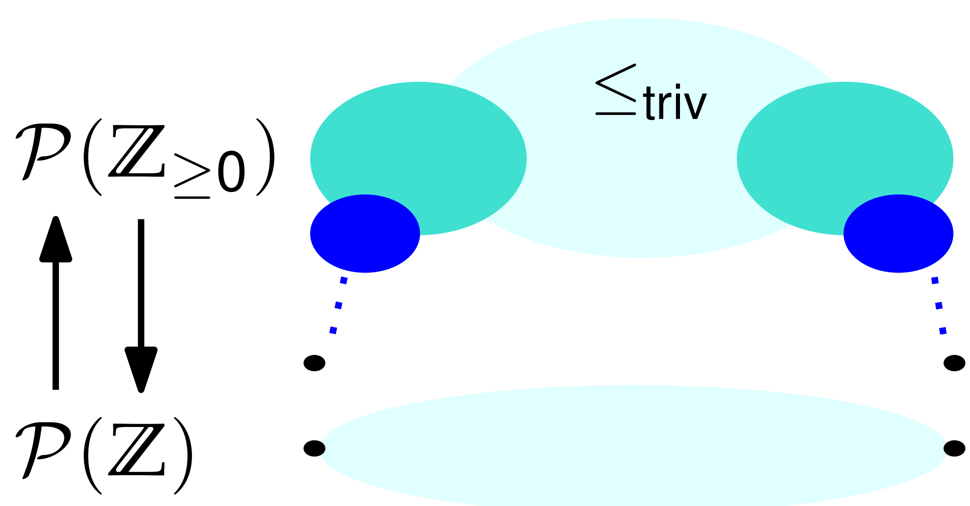
2.a. Groupification

Setting: Π a f.g. commutative monoid, S a f.g. Π-set.

Groupifying
$\Pi \to \mathrm{Gr}(\Pi)$: formally invert all elements (e.g. $\mathbb{Z}^n_{\geq 0} \to \mathbb{Z}^n$)
$S \to \mathrm{Gr}(S)$: universal Π-equivariant map to a $\mathrm{Gr}(\Pi)$-set

Proposition: The pull-back $\mathcal{P}(\mathrm{Gr}(S)) \to \mathcal{P}(S)$ is a homeomorphism with its image, a retract of $\mathcal{P}(S)$ via $\leq \mapsto \leq'$, the *coarsening* of $\leq$ defined by $s \leq' t \Leftrightarrow \exists u \in \Pi : us \leq ut$.

Example
$\Pi = S = \mathbb{Z}_{\geq 0}$
$\mathrm{Gr}(\Pi) = \mathrm{Gr}(S) = \mathbb{Z}$



2.b. The asymptotic range

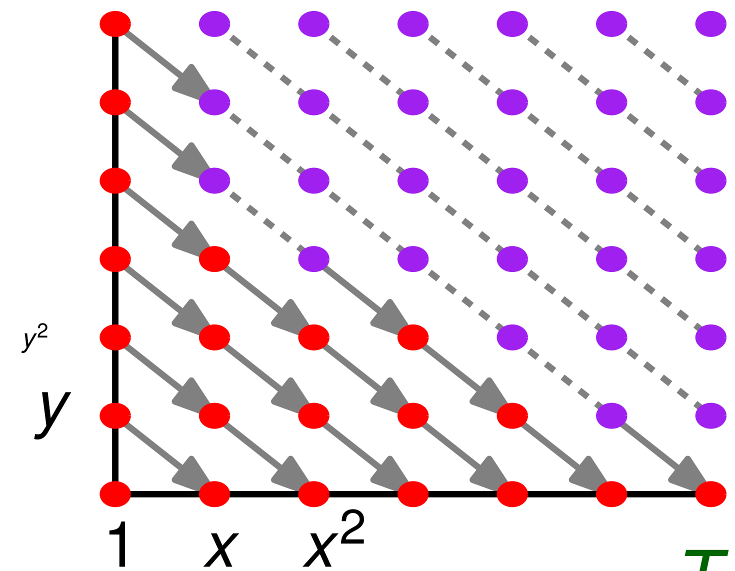
Example: $\leq$ on Mon_2

$x^i y^j \leq x^k y^l \Leftrightarrow$

$i + j < k + l$ or

$i + j = k + l$ and $i > k$ or

$i + j = k + l$ and $x^i y^j, \dots, x^k y^l$ purple



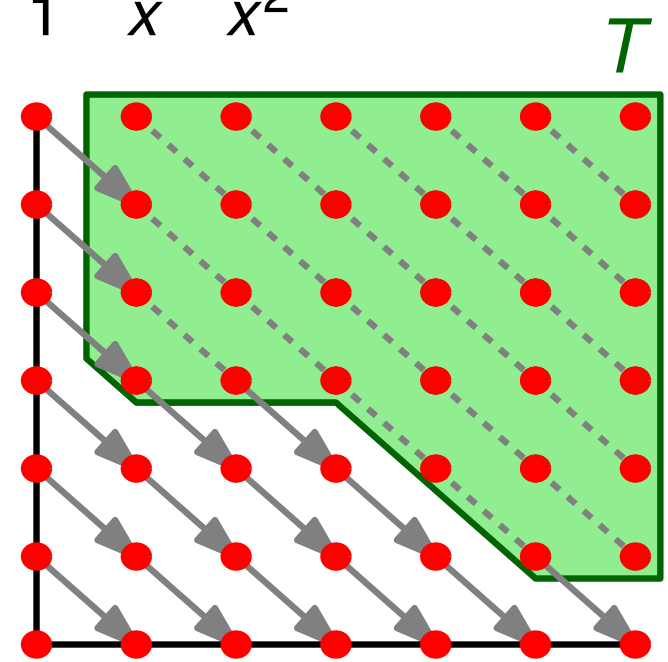
Proposition

$\leq$ on S with coarsening $\leq'$

$\rightsquigarrow \exists \Pi$-stable $T \subseteq S$ s.t.

- $\forall s \in S \exists u \in \Pi : us \in T$
- T intersects each $\approx'$-class

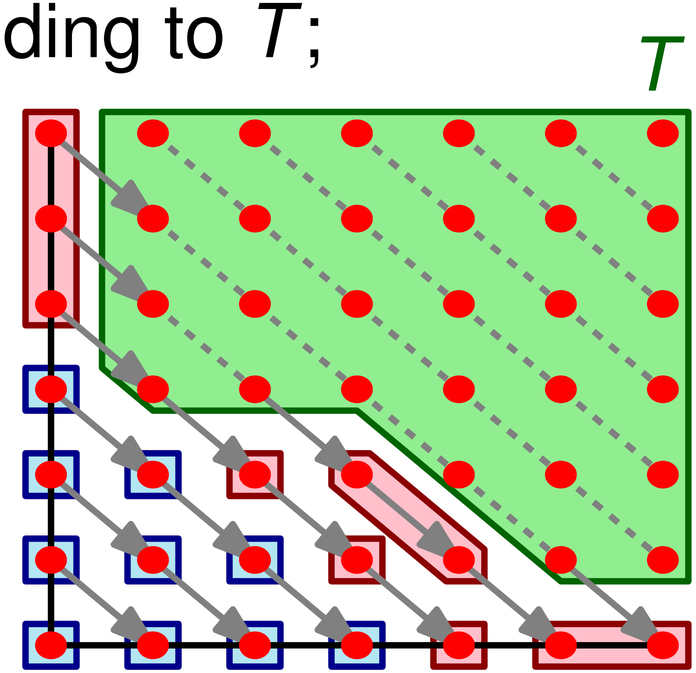
either in $\emptyset$ or in a single $\approx$-class.



T is an *asymptotic range* for $\leq$.

Proposition: S a f.g. Π-set, $\leq$ preorder on S, coarsening $\leq' \rightsquigarrow \exists$ *finite* Π-set X, a Π-equivariant map $\varphi : S \to X$ & a partial order $\preceq$ on X, s.t.:

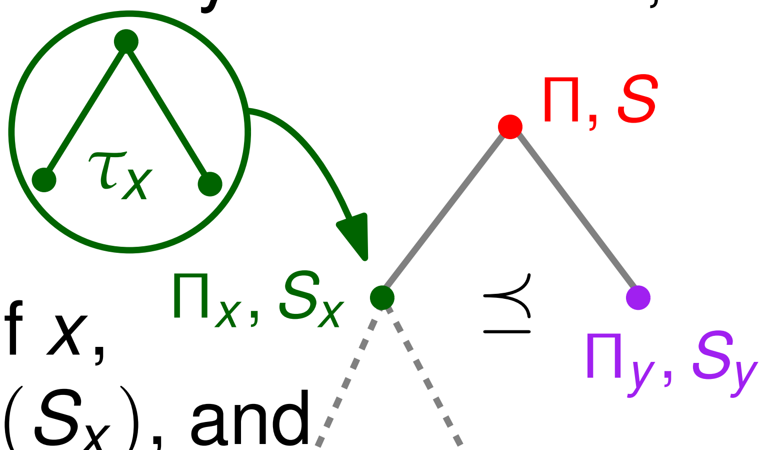
- $s \approx' t \Rightarrow \varphi(s), \varphi(t)$ are comparable w.r.t. $\preceq$;
- each stabiliser Π_x is a f.g. submonoid of Π with strictly fewer generators, except that corresponding to T;
- each $S_x := \varphi^{-1}(x)$ is a f.g. Π_x-set;
- $s \leq t \Leftrightarrow$
$s <' t$ or
$s \approx' t$ and $\varphi(s) \prec \varphi(t)$ or
$s \approx' t$ and $\varphi(s) = \varphi(t) =: x$
and then $s \leq t$ in S_x.



$|X| = 17$

Theorem

Any preorder $\leq$ on the Π-set S is given by a finite tree, in which vertex x is labelled by:



- a f.g. submonoid $\Pi_x \subseteq \Pi$,
- a f.g. Π_x-set $S_x \subseteq S$,
- a partial order $\preceq_x$ on the children of x,
- a preotree τ_x for the $\mathrm{Gr}(\Pi_x)$-set $\mathrm{Gr}(S_x)$, and
- numerical data for τ_x.

Conversely, the locus of numerical data $(p_x)_x$ that gives rise to a preorder on S is a countable union of $\mathbb{Q}$-admissible sets.

$\rightsquigarrow$ **Theorem B**

For any finitely generated commutative monoid Π, there exist $\mathbb{Q}$-admissible sets $A_1 \subseteq \mathbb{R}^{n_1}, A_2 \subseteq \mathbb{R}^{n_2}, \ldots$ and continuous maps $\varphi_i : A_i \to \mathcal{P}(\Pi)$ such that $\mathcal{P}(\Pi) = \bigcup_i \mathrm{im}(\varphi_i)$.

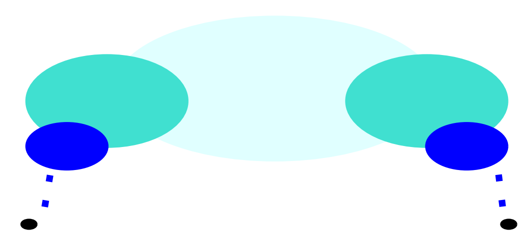
Wrapping up

Question
∃? a preorder on Mon_9 s.t.
$\max\{x_1x_2x_3, x_4x_5x_6, x_7x_8x_9\} < \min\{x_1x_4x_7, x_2x_5x_8, x_3x_6x_9\}$

Our algorithm in Theorem C has not been implemented . . .

Theorem [Bou 15]
There is a *communication ideal* $I \subseteq \mathrm{Mon}_9$ (i.e., the set $\{(I : u) \mid u \in \mathrm{Mon}_9\}$ is totally ordered by inclusion), which contains all mons on the right but none of those on the left.

Now pull back $\subseteq$ along $u \mapsto (I : u) \rightsquigarrow$ *yes!*



Consequence: not all closed points in $\mathcal{P}(\mathrm{Mon}_9)$ are *orders*.
What is the smallest value of 9 for which this holds?

Köszönöm szépen!