1. Symmetric ideals according to Aschenbrenner and Hillar

We will prove the following theorem.

Theorem 1.1. *Let $G = \mathrm{Sym}(\mathbb{N})$ act on the algebra $R = \mathbb{C}[x_0, x_1, \ldots]$ by permutations. Then any G-stable ideal I of R is finitely generated* as G-stable ideal*, that is, there exist finitely many $f_1, \ldots, f_k \in I$ such that I is the smallest G-stable ideal containing $f_1, \ldots, f_k$.*

Background: Hilbert's basis theorem says that *any* ideal in $\mathbb{C}[x_0, \ldots, x_n]$ is finitely generated. But ideals in $\mathbb{C}[x_0, x_1, \ldots]$ need not be. The above theorem says that symmetric ideals are in a sense finitely generated. We say that $\mathbb{C}[x_0, x_1, \ldots]$ is *G-Noetherian*.

The proof is due to Matthias Aschenbrenner and Christopher J. Hillar. They prove something more general, but the main arguments become clear from the proof below.

Definition 1.2. For any map $\pi : \mathbb{N} \to \mathbb{N}$ and $r \in R$ we write πr for the image of r under the homomorphism $R \to R$ sending x_i to $x_{\pi i}$.

Definition 1.3. We define an order $\preceq$ on monomials in $x_0, x_1, \ldots$ as follows: it is the smallest relation on monomials satisfying $1 \preceq 1$ and

$$u \preceq v \Rightarrow u \preceq x_0^b \sigma v \text{ and } x_0^a \sigma(u) \preceq x_0^b \sigma(v)$$

for all u, v and $0 \leq a \leq b$. Here, as in the rest of this talk, $\sigma : \mathbb{N} \to \mathbb{N}, i \mapsto i+1$.

Definition 1.4. For u a monomials we write $|u|$ for the largest i such that x_i appears in u. For $u = 1$ we write $|u| = -\infty$.

Lemma 1.5. *$u \preceq v$ if and only if there is an increasing map $\pi : \{0, \ldots, |u|\} \to \mathbb{N}$ such that πu divides v.*

Proof. The implication $\Rightarrow$ follows by induction: if π does the trick for $u \preceq v$, then $\sigma\pi$, defined on $\{0, \ldots, |u|\}$, does the trick for $u \preceq \sigma v$, and the map defined by

$$i \mapsto \begin{cases} \pi(i-1)+1 & \text{if } i > 0, \text{ and} \\ 0 & \text{if } i = 0 \end{cases}$$

does the trick for $x_0^a u \preceq x_0^b v$.

For the implication $\Leftarrow$, from π one easily reconstructs a sequence of relations that deduce $u \preceq v$ from $1 \preceq 1$. □

Remark 1.6. This lemma implies that $\preceq$ is a partial order.

Proposition 1.7. *The partial order $\preceq$ does not have infinite antichains.*

Proof. Suppose that there do exist infinite antichains. Then there exists an infinite *never-increasing sequence*

$$u_1, u_2, \ldots, u_n, \ldots,$$

that is, a sequence such that $u_i \not\preceq u_j$ for all $i < j$. Moreover, we may take such a sequence with the additional property that $|u_n|$ is minimal among all u_n such that $u_1, \ldots, u_n$ can be extended to an infinite never-increasing sequence.

For all i let a_i be the exponent of x_0 in u_i. Now there exists an infinite sequence $1 \leq i_1 < i_2 < \ldots$ such that

$$a_{i_1} \leq a_{i_2} \leq \ldots$$

(take i_1 such that a_{i_1} is minimal, then take $i_2 > i_1$ such that a_{i_2} is minimal, etc.). But then consider the antichain

$$u_1, \ldots, u_{i_1-1}, u_{i_1}, u_{i_2}, \ldots.$$

Let α be the homomorphism that sends x_{i+1} to x_i for $i \geq 0$ and x_0 to 1. Consider the sequence

$$u_1, \ldots, u_{i_1-1}, \alpha(u_{i_1}), \alpha(u_{i_2}), \ldots$$

By minimality of $|u_{i_1}|$, this sequence is not never-increasing. Hence either there exist $i < i_1$ and $j \geq 1$ such that

$$u_i \preceq \alpha(u_{i_j}),$$

or there exist $1 \leq j \leq k$ such that

$$\alpha(u_{i_j}) \preceq \alpha(u_{i_k}).$$

But in the first case we have

$$u_i \preceq u_{i_j}$$

by the first inductive property of $\preceq$, and in the second case we have

$$u_{i_j} \preceq u_{i_k}$$

by the second inductive property and the fact that $a_{i_j} \leq a_{i_k}$. We thus arrive at a contradiction, hence the proposition is proved. □

Now we can prove the theorem.

Proof of Theorem 1.1. Let I be a G-stable ideal. To any $f \in R$ we associate its *leading monomial* $\text{lm}(f)$ in the lexicographic order, where $x_1 < x_2 < \ldots$. So for instance $x_1^3 < x_1x_2 < x_3$, and x_3 is the leading monomial in $x_1^3 + x_1x_2 + x_3$. Now consider the set M of all $\preceq$-minimal elements of the set $\{\text{lm}(f) \mid f \in I\}$. This is an antichain by definition, hence finite by the proposition. Hence there exist (monic) $f_1, \ldots, f_k \in I$ such that $M = \{\text{lm}(f_1), \ldots, \text{lm}(f_k)\}$. We claim that I equals the smallest G-stable ideal J containing $f_1, \ldots, f_k$.

Indeed, suppose that I contains a (monic) counterexample $f \notin J$. We may assume that $\text{lm}(f)$ lexicographically minimal among counterexamples (since the lexicographic order is a well-order). By construction, there exists an i such that $\text{lm}(f_i) \preceq \text{lm}(f)$. Set $n := |\text{lm}(f_i)|$ and let $\pi : \{1, \ldots, n\} \to \mathbb{N}$ be increasing such that $\pi(\text{lm}(f_i)) | \text{lm}(f)$; say $\text{lm}(f) = u\pi(\text{lm}(f_i))$. Then $\pi(f_i) \in J$ by G-stability, and

$$f' := f - u\pi(f_i) \notin J.$$

We claim that the $\text{lm}(f')$ is lexicographically smaller than $\text{lm}(f)$, contradicting the minimality of the latter. But this is clear from $\text{lm}(\pi(f_i)) = \pi(\text{lm}(f_i))$, so that $\text{lm}(u\pi(f_i)) = u\pi(\text{lm}(f_i)) = \text{lm}(f)$. □

2. G-Noetherianity of some modules

Let the group $G = \text{Sym}(\mathbb{N})$ act on the ring $R = K[y_{ij} | i \neq j]$ by permuting the indices simultaneously. It is easy to see that this ring is not G-Noetherian. However, let $R_{\leq d}$ denote the G-module of polynomials of degree at most d.

Proposition 2.1. *The G-module $R_{\leq d}$ is Noetherian, i.e., every G-submodule of it is finitely generated.*

Proof. We proceed as above: we define two partial orders on monomials in R. The first one has $u \preceq v$ if and only if there is a strictly increasing map $\pi : \{1, \ldots, |u|\} \to \mathbb{N}$ such that $\pi u = v$. Here $|u|$ denotes the maximum among all indices appearing in variables in u. The second order is lexicographic, where the largest index of a variable is most significant, and for definiteness $y_{ij} < y_{ji}$ if $i < j$. So for instance $y_{31} > y_{13} > y_{21}y_{12} > y_{12}^4$.

We claim that the monomials in $R_{\leq d}$ do not contain an infinite antichain with respect to $\preceq$. Indeed, if such an antichain exists, then since there are only finitely many G-orbits of monomials in $R_{\leq d}$, there exists an antichain C contained in some G-orbit. Fix u in this G-orbit for which the indices appearing in its variables are precisely the numbers $1, \ldots, n$. For any element v of Gu construct a monomial $m(v)$ in the variables $x_1, x_2, \ldots$ as follows: let π_v be a bijection from $\{1, \ldots, n\}$ to the set of indices appearing in v, such that $\pi_v u = v$. Then set $m(v) := \pi_v(x_1^1 x_2^2 \cdots x_n^n)$. In particular, if we choose $\pi_u = \mathrm{id}$, then $m(u) = x_1 x_2^2 \cdots x_n^n$. Now m is an injection from Gu to monomials in $x_1, x_2, \ldots$, hence it maps C to an infinite set. This cannot be an antichain in the order $\preceq$ on monomials in the x_i introduced earlier, hence $m(v) \preceq m(w)$ for some $v, w \in C$. Hence there exists an increasing map $\tau : \{1, \ldots, |v|\} \to \mathbb{N}$ such that $\tau m(v) = m(w)$. But then also $\tau v = w$, hence $v \preceq w$.

Now let P be a G-submodule of $R_{\leq d}$. Denote by M the set of all $\preceq$-minimal elements of $\{\mathrm{lm}(f) \mid f \in P\}$. Then M is an antichain, and finite by the above. Hence there exist (monic) $f_1, \ldots, f_k \in M$ such that $M = \{\mathrm{lm}(f_1), \ldots, \mathrm{lm}(f_k)\}$. We claim that P equals the G-module Q generated by the f_i.

Indeed, suppose that P contains a (monic) counterexample $f \notin Q$. We may assume that $\mathrm{lm}(f)$ lexicographically minimal among counterexamples (since the lexicographic order is a well-order). By construction, there exists an i such that $\mathrm{lm}(f_i) \preceq \mathrm{lm}(f)$. Set $n := |\mathrm{lm}(f_i)|$ and let $\tau : \{1, \ldots, n\} \to \mathbb{N}$ be increasing such that $\tau(\mathrm{lm}(f_i)) = \mathrm{lm}(f)$. Then $\tau(f_i) \in Q$ by G-stability, and

$$f' := f - \tau(f_i) \notin Q.$$

We claim that the $\mathrm{lm}(f')$ is lexicographically smaller than $\mathrm{lm}(f)$, contradicting the minimality of the latter. But this is clear from $\mathrm{lm}(\tau(f_i)) = \tau(\mathrm{lm}(f_i)) = \mathrm{lm}(f)$. $\square$