Subranks of bilinear maps

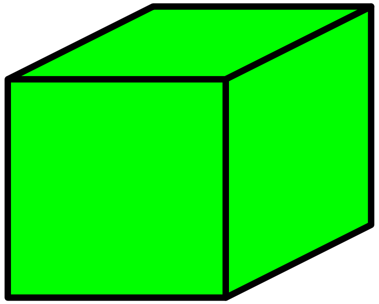
(Joint with Biaggi-Chang-Rupniewski
and with Biaggi-Eggleston)

Jan Draisma
University of Bern

Subrank

$$U \times V \xrightarrow{\quad f \quad} W$$

$f :$



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$$\begin{array}{ccc} U \times V & \xrightarrow{\quad f \quad} & W \\ \alpha \times \beta \uparrow & & \downarrow \gamma \\ K^r \times K^r & & K^r \end{array}$$

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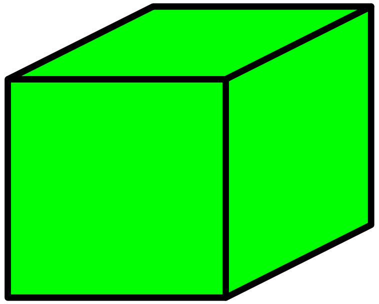

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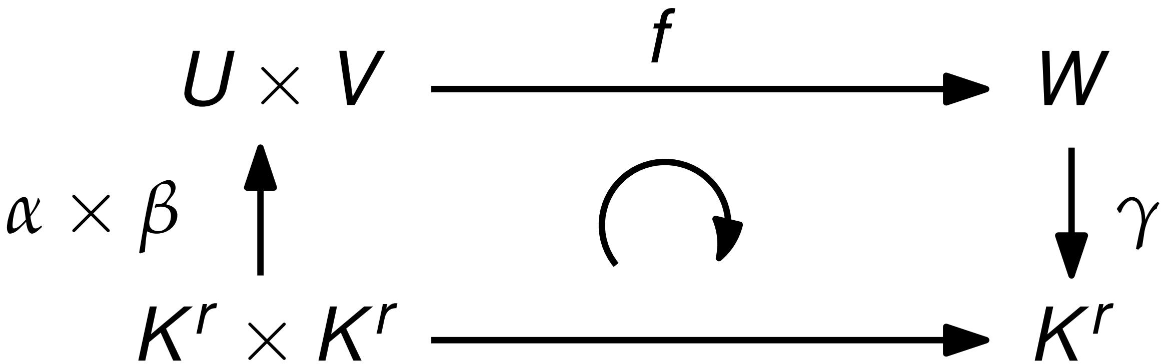

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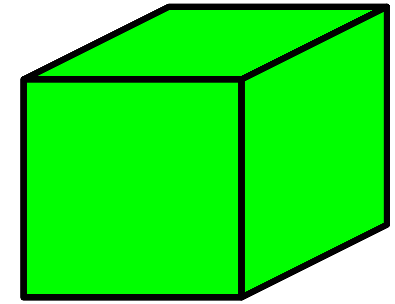
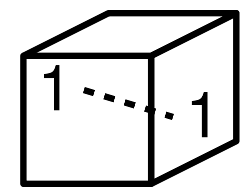
$f :$ 

$I_r :$ 

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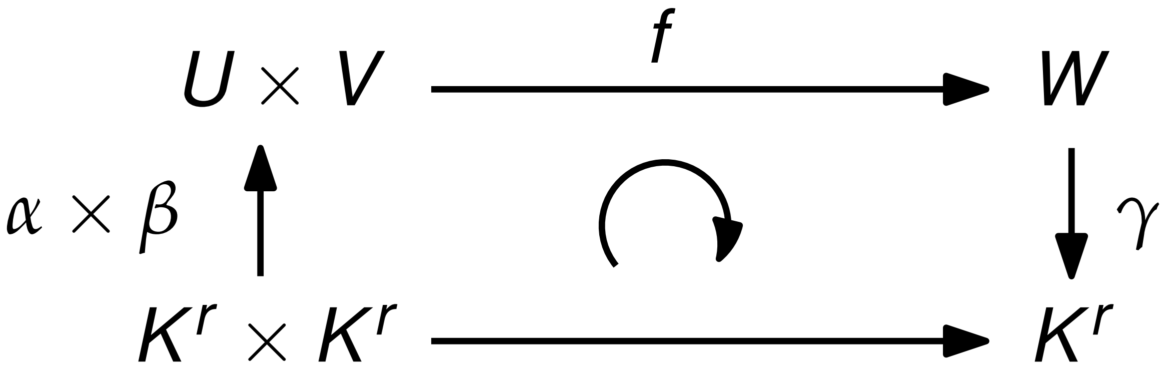

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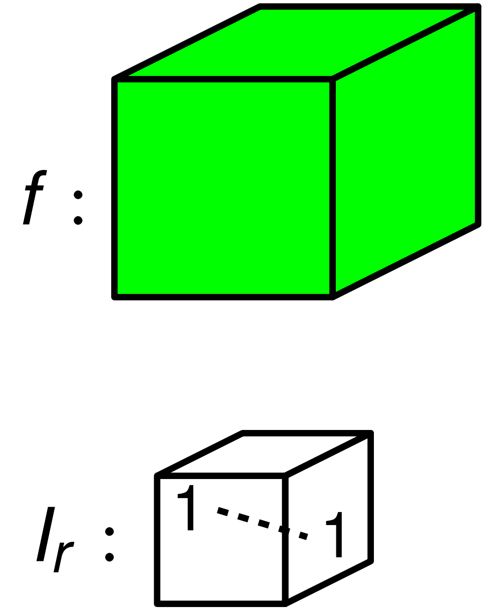

Definition. [Strassen]

The *subrank* of f is $Q(f) := \max\{r \mid \exists \alpha, \beta, \gamma : \gamma \circ f \circ (\alpha \times \beta)$ is the Hadamard product $I_r : K^r \times K^r \to K^r\}$.

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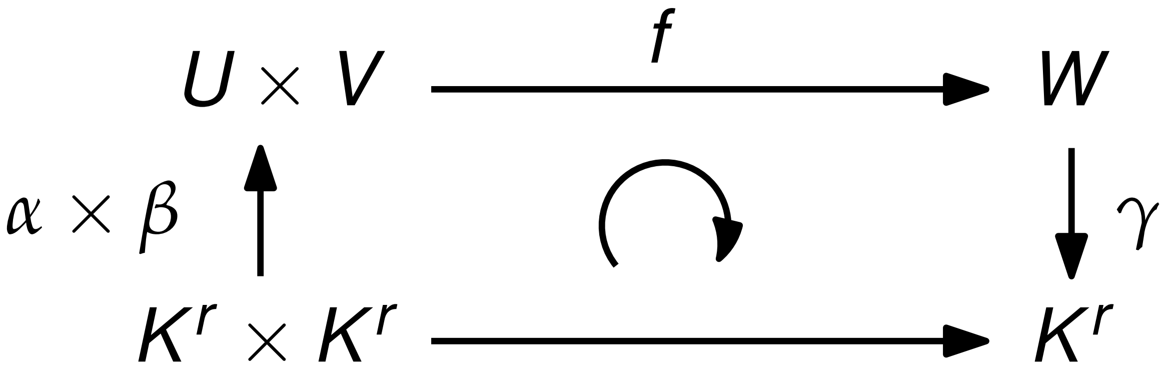

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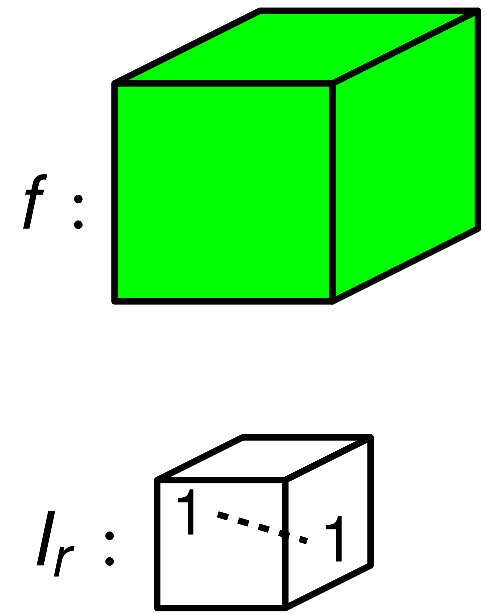

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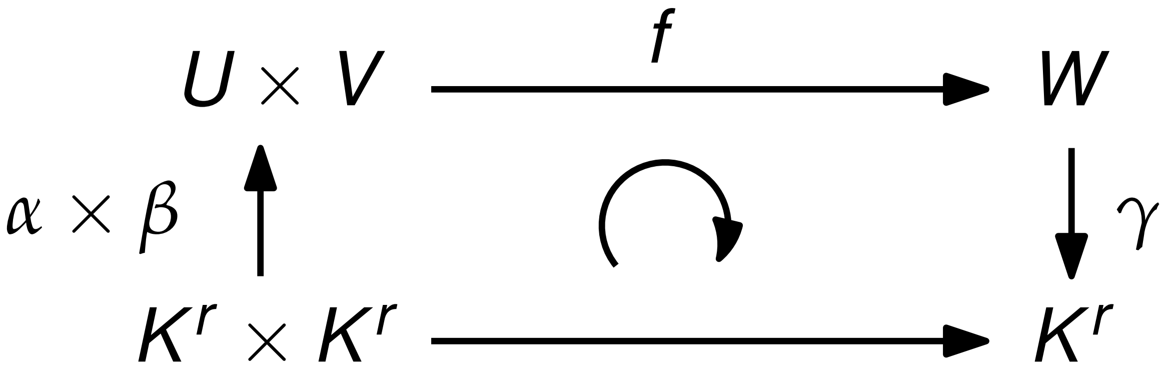

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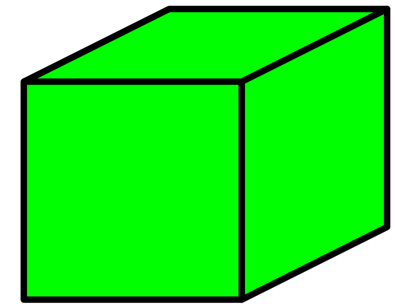

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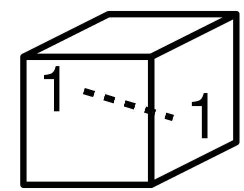

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Remark. α and β are injective, γ is surjective, so $Q(f) \leq \min\{\dim(U), \dim(V), \dim(W)\}$.

Example. $Q(f) = 0 \Leftrightarrow f = 0$, and $Q(I_r) = r$.

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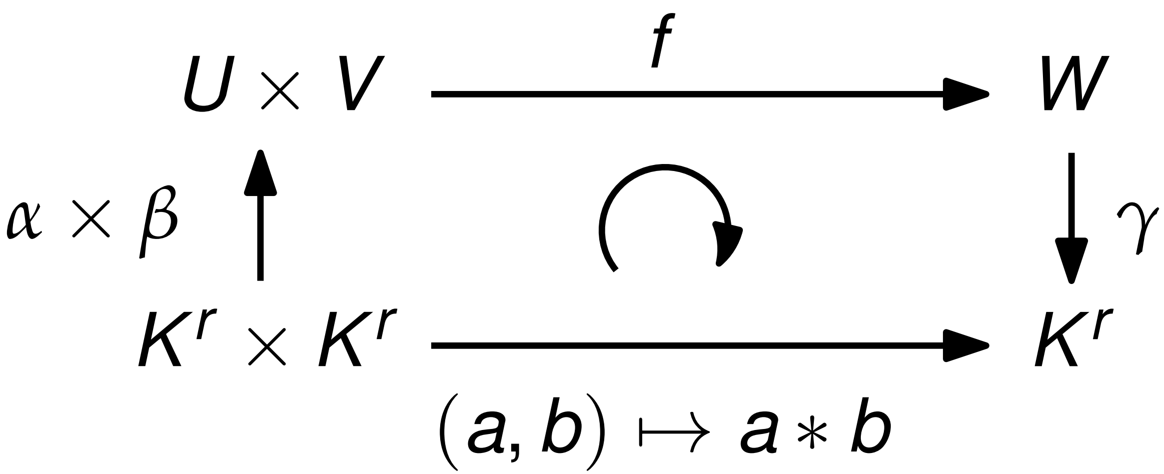

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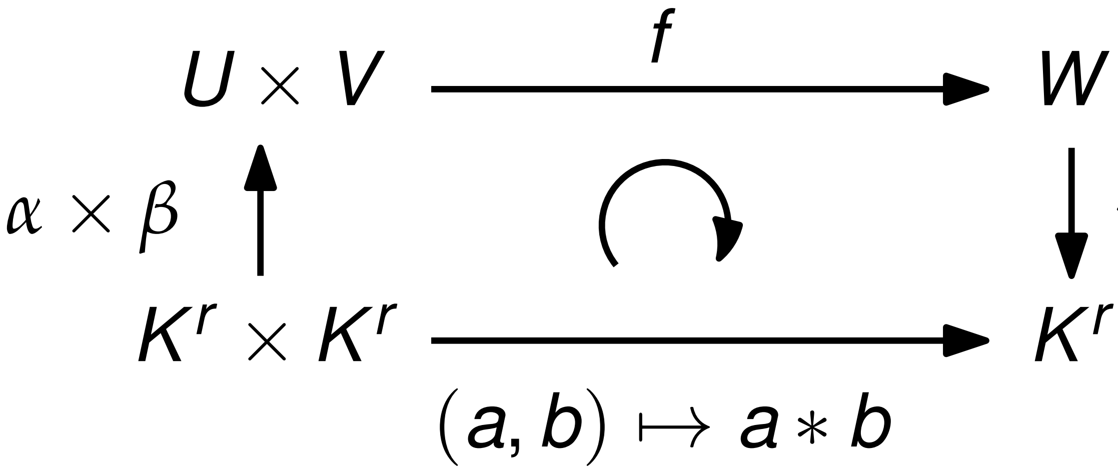

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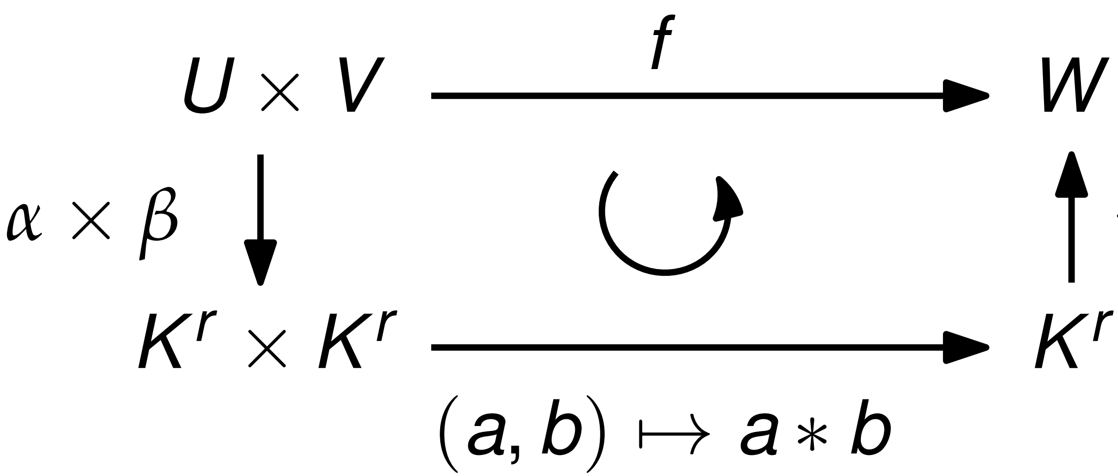

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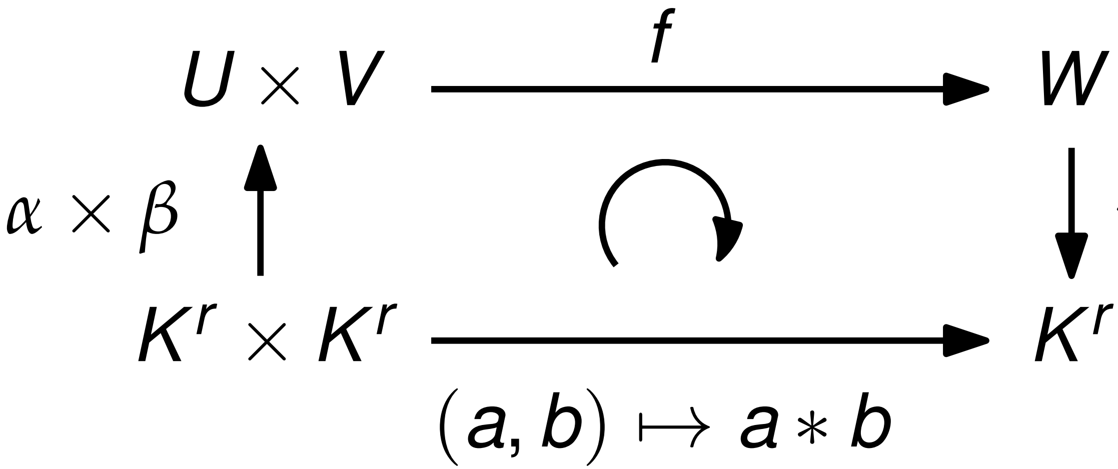

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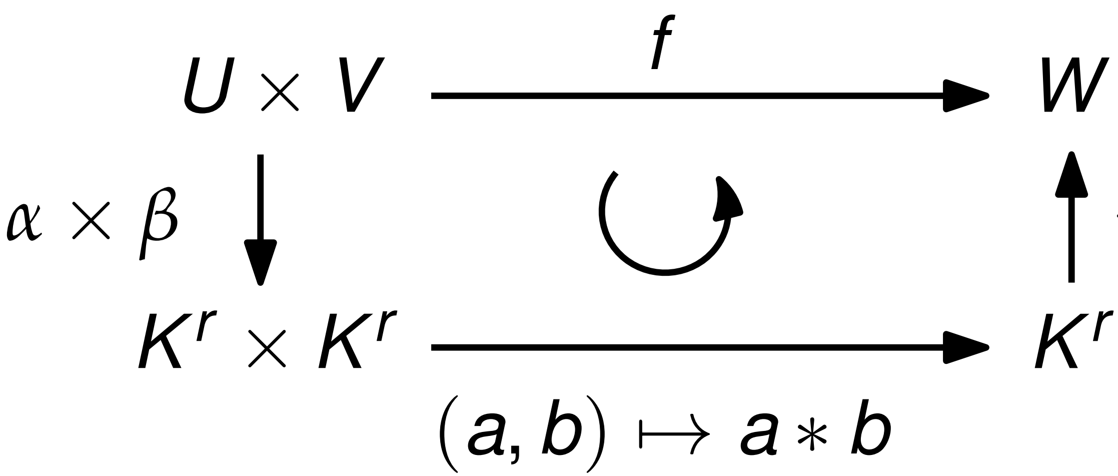

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Remark. If f *concise* (surjective, zero radical), then

$$Q(f) \leq \min\{\dim(U), \dim(V), \dim(W)\}$$
$$\leq \max\{\dim(U), \dim(V), \dim(W)\} \leq R(f).$$

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$f : U \times V \to W \rightsquigarrow f_L : U_L \times V_L \to W_L$
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Example. $K = \mathbb{R}$, $U = V = W = \mathbb{C}$ as $\mathbb{R}$-space,
$f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ scalar multiplication as $\mathbb{R}$-bilinear map.

Then $1 \le Q_{\mathbb{R}}(f) \le 2$ and in fact $Q(f) = 1$ since $f(u, v) \ne 0$ when $u, v \ne 0$. But $Q_{\mathbb{C}}(f_{\mathbb{C}}) = 2$ due to $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$.

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Remark • $Q(f) \leq \underline{Q}(f) \leq \min\{\dim(U), \dim(V), \dim(W)\}$
• $\{f \mid \underline{Q}(f) \leq r\}$ is typically not closed!

Generic subrank

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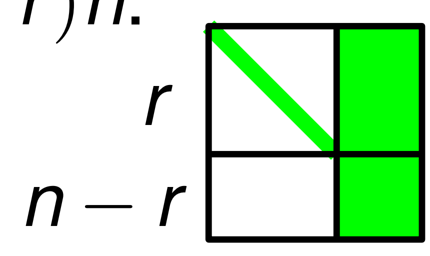
• The space of such $\widetilde{f}$ has dim $n^3 - r^3$
and is stabilised by a group of dim $2r + 3(n-r)n$.



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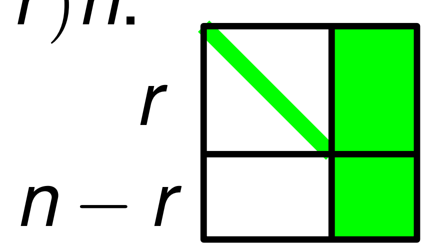

• So $\{f \mid Q(f) \geq r\}$ has dim
$\leq n^3 - r^3 + 3n^2 - 2r - 3(n-r)n$
$= n^3 - r(r^2 - 3n + 2) \rightsquigarrow$ density requires $r \leq \sqrt{3n-2}$. $\square$

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Let a reductive group G acts on an affine variety X and let $p, q \in X$ with $q \in \overline{Gp}$. Then $\exists \widetilde{q} \in Gq$ and $\lambda : K^* \to G$:

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- Take $X := U \otimes V \otimes W$ and $p := f$ (general) and $q := I_r \rightsquigarrow$ decompose $X = X_{<0} \oplus X_0 \oplus X_{>0}$ using λ.

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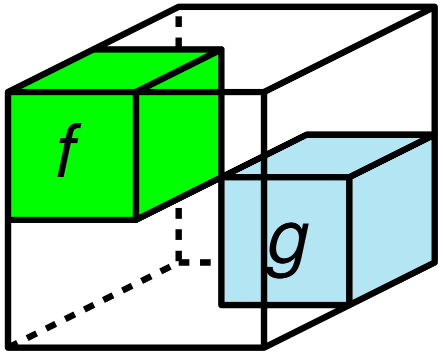
- For $\geq$, we show $\exists \lambda : \lim_{t\to 0} \lambda(t)f \in \mathrm{GL}_n^3 \cdot I_r$, $r \sim 2\sqrt{n}$. $\square$

Non-additivity

Definition. $f : U \times V \to W, g : A \times B \to C \rightsquigarrow f \oplus g : (U \oplus A) \times (V \oplus B) \to (W \oplus C)$ is their *direct sum*.



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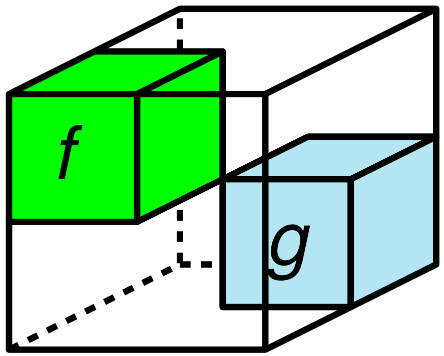

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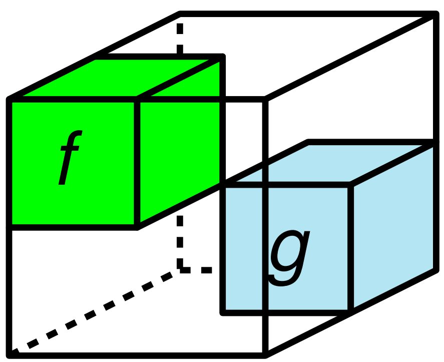

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Observation [Derksen-Makam-Zuiddam]

This can be (very) strict! Take $f : K^n \times K^n \to K^n$ general, so that $g := I_n - f$ is also general. Then $2n \geq \underline{Q}(f \oplus g) \geq \underline{Q}(f + g) = n$ but $Q(f) + Q(g) \approx 2\sqrt{3n}$.

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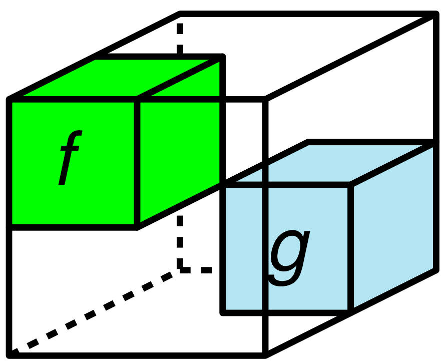

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Same applies to $\underline{Q}$, just with $2C\sqrt{n}$ from B-C-D-R.

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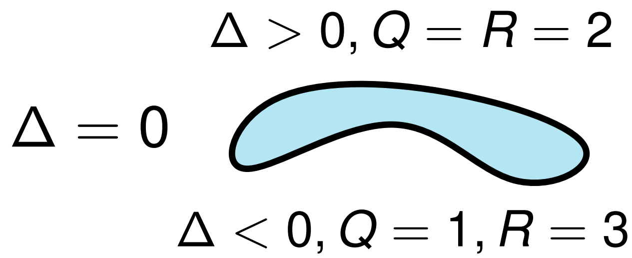

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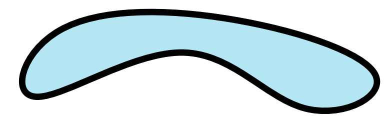

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For $U = V = W = \mathbb{R}^4$ the typical ranks are $2, 3$.

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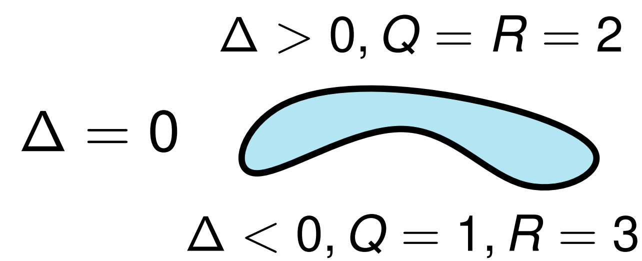

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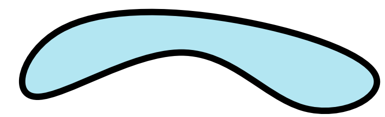

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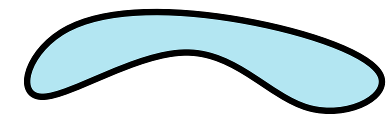

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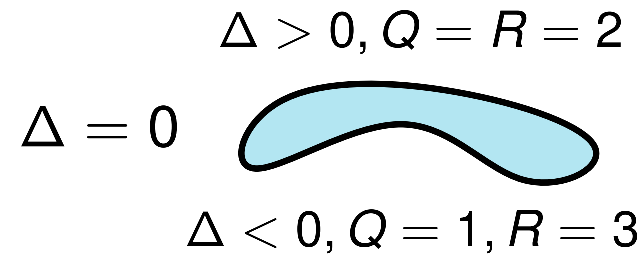

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Theorem. $\{$typical subranks$\}$ form an interval (uses Bernardi-Blekherman-Ottaviani).

Real versus complex subrank

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Proof is a nontrivial induction on n; for an arbitrary division algebra A over K we get $Q(\mu_{A^n}) \leq \frac{\dim_K(A)}{2} \cdot n$.

Some open questions

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Thank you!

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